Special Relativity and Electromagnetism - 6CCM331A

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Chapter 1

Index Notations

1.1 Matrix multiplication in index notation

It is easy to see that the multiplication of two 2×2 matrices A, B together gives the identity

$$(AB)_{ik} = \sum_{j=1}^{2} A_{ij} B_{jk}$$

Convention 1.1.1. *Given an expression involving indices we use the following conventions*

- *If an index appears once on the right hand side, it must appear only once on the left. These are referred to as* ***free indices****.*
- *If an index appears twice on either side, it is summed over. Such indices are referred to as* ***dummy indices****.*

This is referred to as the ***Einstein summation convention****.*

Remark. *Any dummy index can be replaced by another letter without affecting the truth of the expression. The same can not be said of free indices.*

Example 1.1.2. *Consider again the case of matrix multiplication of A and B. Then we have the following notation for their multiplication*

$$(AB)_{ik} = A_{ij} B_{jk}$$

where the free indices are i and k. The dummy index j is summed over.

Example 1.1.3. *Let A be an $n \times m$ matrix and v an m-dimensional vector. Then*

$$(Av)_i = A_{ij} v_j$$

1.2 General conventions and notation

Convention 1.2.1. *We denote the three spacial coordinates of a vector $\vec{x}$ by x_i where i takes the values $1, 2, 3$. We also write $\vec{x} = (x_1, x_2, x_3)$.*

Convention 1.2.2. *Derivatives with respect to the three spacial coordinates of a vector are written in the following ways*

$$\vec{\nabla} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) = (\partial_1, \partial_2, \partial_3)$$

Definition 1.2.3. *We denote the* ***Kronecker delta*** *tensor δ_{ij} to be the following matrix*

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{ij}$$

Remark. *The Kroneker delta is a symmetric tensor. That is to say that*

$$\delta_{ij} = \delta_{ji}$$

for all $i = 1, 2, 3$ and $j = 1, 2, 3$.

Property 1.2.4. *Let $A_{\ldots}$ be an index expression. The Kronecker delta satisfies the following two identities*

$$A_{\ldots i \ldots} \delta_{ik} = A_{\ldots k \ldots}$$
$$\delta_{ii} = 3$$

Definition 1.2.5. *We denote the* ***epsilon*** *tensor (also known as the* ***Levi-Civita*** *symbol) to be the one satisfying the following expressions*

$$\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = 1$$
$$\epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1$$

It is zero otherwise.

Remark. *We note that the epsilon tensor is anti-symmetric. Interchanging any two indices causes a change of sign:*

$$\epsilon_{ijk} = -\epsilon_{ikj} = \epsilon_{kij} = -\epsilon_{jki} = -\epsilon_{jik}$$

for all $i, j, k = 1, 2, 3$.

Property 1.2.6. *The epsilon tensor and Kronecker delta sastify the following identities*

$$\begin{aligned}
&\epsilon_{ijk}\epsilon_{inm} = \delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn} \\
&\epsilon_{ijk}\epsilon_{ijm} = 2\delta_{km} \\
&\epsilon_{ijk}\epsilon_{ijk} = 6
\end{aligned}$$

Definition 1.2.7. *We can define the* ***dot product*** *of two vectors* $\vec{a}$ *and* $\vec{b}$ *in terms of indices:*

$$\vec{a} \cdot \vec{b} = a_i b_i$$

Definition 1.2.8. *We can define the* ***cross product*** *of two vectors* $\vec{a}$ *and* $\vec{b}$ *in terms of indices:*

$$(\vec{a} \times \vec{b})_i = \epsilon_{ijk} a_j b_k$$

Example 1.2.9. *Using the above definitions and identities, we can prove the following*

$$\begin{aligned}
(\vec{a} \times (\vec{b} \times \vec{c}))_i &= \epsilon_{ijk} a_j (\vec{b} \times \vec{c})_k \\
&= \epsilon_{ijk} a_j (\epsilon_{knm} b_n c_m) \\
&= \epsilon_{ijk}\epsilon_{knm} a_j b_n c_m \\
&= \epsilon_{kij}\epsilon_{knm} a_j b_n c_m \\
&= (\delta_{in}\delta_{jm} - \delta_{im}\delta_{jn}) a_j b_n c_m \\
&= \delta_{in}\delta_{jm} a_j b_n c_m - \delta_{im}\delta_{jn} a_j b_n c_m \\
&= \delta_{jm} a_j b_i c_m - \delta_{jn} a_j b_n c_i \\
&= a_j b_i c_j - a_n b_n c_i \\
&= b_i(\vec{a} \cdot \vec{c}) - c_i(\vec{a} \cdot \vec{b})
\end{aligned}$$

Example 1.2.10. *Let* $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ *be the length of the vector* $\vec{x}$. *We can derive an identity for its derivative as follows:*

$$\begin{aligned} \partial_i r &= \frac{1}{2} \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \partial_i (x_1^2 + x_2^2 + x_3^2) \\ &= \frac{1}{2r} \cdot 2 \cdot x_i \\ &= \frac{x_i}{r} \end{aligned}$$

Proposition 1.2.11. *Let* ϕ *be a scalar function. Then* $curl(\nabla\phi) = 0$.

Proof. We have that

$$\begin{aligned} (curl(\nabla\phi))_i &= (\vec{\nabla} \times \nabla\phi)_i \\ &= \epsilon_{ijk} \partial_j (\nabla\phi)_k \\ &= \epsilon_{ijk} \partial_j \partial_k \phi \\ &= \epsilon_{ikj} \partial_k \partial_j \phi \\ &= -\epsilon_{ijk} \partial_j \partial_k \phi \\ &= 0 \end{aligned}$$

□

Proposition 1.2.12. *Let* $\vec{a}$ *be a vector. Then* $div(curl(\vec{a})) = 0$

Proof. We have that

$$\begin{aligned} div(curl(\vec{a})) &= \partial_i (curl(\vec{a}))_i \\ &= \partial_i \epsilon_{imn} \partial_m a_n \\ &= \epsilon_{imn} \partial_i \partial_m a_n \\ &= \epsilon_{min} \partial_m \partial_i a_n \\ &= -\epsilon_{imn} \partial_i \partial_m a_n \\ &= 0 \end{aligned}$$

□

Chapter 2

Maxwell's Equations

2.1 Charge, Charge Density, Current Density

Observation 2.1.1. *From experiments, we can observe the following*

- *Physical objects can carry* ***electric charge*** *which is usually denoted by q or Q. We can build devices that measure the electric charge carried by a body.*
- *The electric charge of a body is independent of other fundamental properties such as mass. We therefore require a seperate unit to measure it which is called a* ***Coulomb*** *and is denoted by C.*
- *Electric charge is additive. If we bring two bodies, of charges q_1 and q_2, together their total charge is $q_1 + q_2$. Through this, we observe that electric charge can be either positive or negative. For any process, the total charge of all bodies involved is conserved.*

Definition 2.1.2. *Assume that a system consists of charge that is distributed continuously in space and time. We write $\rho(\vec{x}, t)$ for the* ***charge density*** *at a point $\vec{x} \in \mathbb{R}^3$ and time $t \in \mathbb{R}$.*

Remark. *Given a charge density $\rho(\vec{x}, t)$, we can calculate the total charge contained in a volume V at a given time t by*

$$Q = \int_V \rho(\vec{x}, t) \, d^3x$$

Definition 2.1.3. *Assume that a system consists of charged particles distributed continuously in space and time with a charge density $\rho(\vec{x}, t)$. Furthermore, let each point in space and time have a velocity $\vec{v}(\vec{x}, t)$. We define the* ***current density*** *to be*

$$\begin{aligned} \vec{j}(\vec{x}, t) &= \begin{pmatrix} \rho(\vec{x}, t) v_1(\vec{x}, t) \\ \rho(\vec{x}, t) v_2(\vec{x}, t) \\ \rho(\vec{x}, t) v_2(\vec{x}, t) \end{pmatrix} \\ &= \rho(\vec{x}, t) \vec{v}(\vec{x}, t) \end{aligned}$$

with units $\frac{C}{sm^2} = \frac{A}{m^2}$ where A is an Ampere.

2.2 Electric and Magnetic Fields

Observation 2.2.1. *Consider two static point-like charges with spatial coordinates $\vec{x}_1$ and $\vec{x}_2$. Then the force acting between them is*

$$\vec{F}_{q_1 q_2} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_{12}^2} \frac{\vec{x}_2 - \vec{x}_1}{r_{12}}$$

where r_{12} is the absolute distance between them and $\varepsilon_0 = 8.85419 \times 10^{-12} \frac{C^2}{m^2 N}$ is the ***vacuum permittivity of free space****.*

Observation 2.2.2. *Consider a system of charges and a* ***probe charge*** *q which is an extra point-like charge that we can move at will through the space. We observe that the force acting on the probe charge is proportional to q:*

$$\vec{E}(\vec{x}, t) = \frac{1}{q} \vec{F}(\vec{x}, t)$$

$\vec{E}$ is a vector field and is referred to as the ***electric field****.*

Observation 2.2.3. *Consider a system of charges and a probe charge moving through the system with velocity $\vec{v}$. In addition to the force mentioned above, we observe another force acting on the charge which is proportional to its speed $|\vec{v}|$ and orthogonal to the direction of the velocity:*

$$\vec{F}(\vec{x}, t) = q\vec{v} \times \vec{B}(\vec{x}, t)$$

The quantity $\vec{B}$ is a vector field and is referred to as the ***magnetic field****.*

Definition 2.2.4. *Consider a system of charges containing a charge moving with some velocity $\vec{v}$. Then the force acting on it is*

$$\vec{F}(\vec{x},t) = q\left(\vec{E}(\vec{x},t) + \vec{v}(\vec{x},t) \times \vec{B}(\vec{x},t)\right)$$

and is referred to as the ***Lorentz Force****.*

2.3 Surface and Line Integrals

Definition 2.3.1. *We define the* ***volume integral*** *of some function $f(\vec{x})$ to be the integral over the interior of some volume V:*

$$\int_V f(\vec{x})\ d^3x$$

Example 2.3.2. *The integral of the charge density over some volume V is a volume integral*

$$Q_V(t) = \int_V \rho(\vec{x},t)\ d^3x$$

Definition 2.3.3. *We define the* ***surface integral*** *of some function $f(\vec{x})$ to be the integral over some surface S:*

$$\int_S f(\vec{x}) \cdot d\vec{S}$$

where $d\vec{S}$ is the infinitesimal surface element at the point $\vec{x}, t$ given by

$$d\vec{S} = \vec{n}(\vec{x},t)dS$$

$\vec{n}(\vec{x},t)$ is the unit normal vector at each point $\vec{x}, t$.

Definition 2.3.4. *Let $\vec{K} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field and $\vec{\gamma} : [a,b] \to \mathbb{R}^3$ a curve. Let $C \in \mathbb{R}^3$ be the image of the interval $[a,b]$ under $\vec{\gamma}$. Then we define the line integral of $\vec{K}$ over C to be*

$$\int_C \vec{K} \cdot d\vec{x} = \int_a^b \vec{K}(\vec{\gamma}(s)) \cdot \frac{\partial \vec{\gamma}}{\partial s} ds$$

Remark. *The line integral is invariant under reparametrization of the curve. If $a = b$ then the curve is closed and the integral is denoted by*

$$\oint_\gamma \vec{K} \cdot d\vec{x}$$

Example 2.3.5. *Consider the point $\vec{x} \in \mathbb{R}^3$ and let $r = |\vec{x}|$. Let $\vec{K}(\vec{x}) = f(r)\vec{x}$ be a vector field and $\gamma = (cos(s), sin(s), 0)$ where $0 \leq s \leq 2\pi$. We have that*

$$\begin{aligned}
\oint_\gamma \vec{K} \cdot d\vec{x} &= \int_0^{2\pi} \vec{K}(\gamma(s)) \cdot \frac{\partial \vec{\gamma}}{\partial s} \, ds \\
&= \int_0^{2\pi} f\left(\sqrt{cos^2(s) + sin^2(s)}\right) (cos(s), sin(s), 0) \cdot (-sin(s), cos(s), 0) \, ds \\
&= f(1) \int_0^{2\pi} (-cos(s)sin(s) + sin(s)cos(s)) \, ds \\
&= 0
\end{aligned}$$

Example 2.3.6. *Consider the point $\vec{x} \in \mathbb{R}^3$ and let $r = |\vec{x}|$. Let $\vec{K}(\vec{x}) = f(r)(y, -x, 0)$ be a vector field and $\gamma = (cos(s), sin(s), 0)$ where $0 \leq s \leq 2\pi$. We have that*

$$\begin{aligned}
\oint_\gamma \vec{K} \cdot d\vec{x} &= \int_0^{2\pi} \vec{K}(\gamma(s)) \cdot \frac{\partial \vec{\gamma}}{\partial s} \, ds \\
&= \int_0^{2\pi} f\left(\sqrt{cos^2(s) + sin^2(s)}\right) (sin(s), -cos(s), 0) \cdot (-sin(s), cos(s), 0) \, ds \\
&= f(1) \int_0^{2\pi} (-sin^2(s) - cos^2(s)) \, ds \\
&= -f(1) \int_0^{2\pi} ds \\
&= -2\pi f(1)
\end{aligned}$$

2.4 Stoke's Theorem

Theorem 2.4.1. *(Fundamental Theorem of Calculus) Let $a, b \in \mathbb{R}$ where $a < b$. Let $f(x) : \mathbb{R} \to \mathbb{R}$ be a function and $f'(x)$ its derivative. Then*

$$\int_a^b f'(x) \, dx = f(b) - f(a)$$

Proposition 2.4.2. *Let $\phi : \mathbb{R}^m \to \mathbb{R}$ be a scalar function and $\vec{\gamma}(s)$ a curve between two points a and b. Let $\vec{a} = \vec{\gamma}(a)$ and $\vec{b} = \vec{\gamma}(b)$. Then*

$$\int_\gamma \vec{\nabla}\phi \cdot d\vec{x} = \phi(\vec{b}) - \phi(\vec{a})$$

Proof. We have that

$$\begin{aligned}\int_\gamma \vec{\nabla}\phi \cdot d\vec{x} &= \int_\gamma \partial_i \phi(\vec{\gamma}(s)) \frac{\partial \gamma_i(s)}{\partial s}\, ds \\ &= \int_a^b \phi'(\vec{\gamma}(s))\, ds \\ &= \phi(\vec{\gamma}(b)) - \phi(\vec{\gamma}(a)) \\ &= \phi(\vec{b}) - \phi(\vec{a})\end{aligned}$$

□

Theorem 2.4.3. *(Stoke's Theorem)*
Let $\vec{K}$ be a vector field and S a surface. Then

$$\int_S (\vec{\nabla} \times \vec{K}) \cdot\, d\vec{S} = \oint_{\partial S} \vec{K} \cdot d\vec{x}$$

where ∂S is the curve given by the boundary of the surface. The direction of the curve should be chosen depending on the orientation of the unit normal vector.

Theorem 2.4.4. *(Divergence Theorem)*
Let $\vec{K}$ be a vector field and V a volume. Then

$$\int_V (\vec{\nabla} \cdot \vec{K})\, d^3x = \oint_{S=\partial V} \vec{K} \cdot d\vec{S}$$

where ∂V is the surface given by the boundary of the volume. The unit normal vector to the surface must point outside the integration region.

2.5 Maxwell's Equations

Observation 2.5.1. *Maxwell observed that given a system of charges the charge density* ρ*, current density* $\vec{j}$ *and corresponding electric field* $\vec{E}$ *and magnetic field* $\vec{B}$ *satisfy the following equations*

- $\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$
- $\vec{\nabla} \times \vec{B} = \frac{1}{c^2}\partial_t \vec{E} + \frac{1}{c^2 \varepsilon_0}\vec{j}$
- $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$
- $\vec{\nabla} \cdot \vec{B} = 0$

where c *is the speed of light.*

2.6 The Continuity Equation

Proposition 2.6.1. *Consider a system of charges. Then the charge density* ρ *and current density* $\vec{j}$ *satisfy the following equation:*

$$\partial_t \rho = -\vec{\nabla} \cdot \vec{j}$$

Proof. This follows from the Maxwell Equations. M2 and M3 can be written in the forms

$$\vec{j} = c^2 \varepsilon_0 \left(\vec{\nabla} \times \vec{B}\right) - \varepsilon_0 \partial_t \vec{E}$$
$$\rho = \varepsilon_0 \left(\vec{\nabla} \cdot \vec{E}\right)$$

It follows that

$$\vec{\nabla} \cdot \vec{j} = \varepsilon_0 c^2 \vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{B}\right) - \varepsilon \partial_t \vec{\nabla} \cdot \vec{E} \tag{2.1}$$
$$\partial_t \rho = \varepsilon_0 \partial_t \vec{\nabla} \cdot \vec{E} \tag{2.2}$$

Since the divergence of the curl of a vector vanishes, the first term of Equation (2.1) equals to 0. The proposition follows by adding Equations (2.1) and (2.2). □

Remark. *The above equation is referred to as the* ***continuity equation****. It says that if charge is moving out of a volume then the amount of charge in the volume will decrease so that the rate of change of charge is negative.*

2.7 Integral Form of Maxwell's Equations

Proposition 2.7.1. *Consider the 3rd Maxwell equation $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$. Then it takes the following 'integral form':*

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\varepsilon_0}$$

where S is a closed surface and Q is the charge contained in S

Proof. We start by integrating both sides of M3 over some volume V:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$
$$\int_V \vec{\nabla} \cdot \vec{E} \; d^3x = \int_V \frac{\rho}{\varepsilon_0} \; d^3x$$

We first note that the right hand side is exactly the amount of charge contained inside the volume V. We can then apply the divergence theorem to the left hand side to get

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{Q_V}{\varepsilon_0}$$

where S is the surface given by the boundary of the volume V and Q_V is the charge contained inside V. □

Remark. *The above equivalence is referred to as* ***Gauss' Law****.*

Example 2.7.2. *Consider a point-like charge at the origin $\vec{x} = 0$. Since the charge has no distinguishable direction, its resulting electric field $\vec{E}$ should be spherically symmetric. This implies that $|\vec{E}(\vec{x})| = |\vec{E}(\vec{x}')|$ when $|\vec{x}| = |\vec{x}'|$. The force acting on any probe charge should be directed towards or away from the origin which implies that $\vec{E}$ is directed along $\vec{x}$. We can, therefore, assume that the electric field is of the form*

$$\vec{E} = f(r)\frac{\vec{x}}{r}$$

where $r = |\vec{x}|$. Using Gauss' Law and a sphere of radius r as the surface S with unit normal vector $\vec{n}$. we get

$$\begin{aligned} \frac{Q}{\varepsilon_0} &= \oint_S \vec{E} \cdot d\vec{S} \\ &= \oint_S \vec{E} \cdot \vec{n} \; dS \end{aligned}$$

Now since the electric field points in the same direction as the normal vector of the sphere, we have that

$$\begin{aligned}\oint_S \vec{E}\cdot\vec{n}\,ds &= \oint_S |\vec{E}||\vec{n}|cos(0)\,ds\\ &= \oint_S |\vec{E}|\,ds\\ &= |\vec{E}|\oint_S ds\\ &= 4\pi|\vec{E}|r^2\\ &= 4\pi f(r)r^2\end{aligned}$$

Inserting this back into the equation for the electric field, we get

$$\vec{E}(r) = \frac{Q\vec{x}}{4\pi\varepsilon_0 r^3}$$

Example 2.7.3. *Consider a uniformly charged plane over the xy plane. To represent charge we use the charge per surface area* $\sigma = \frac{Q}{A}$*. We want to find the electric field at some point of vertical distance y from the plane.*
Consider a unit cylinder whose top intersects the points $(0, y, 0)$ *and* $(0, -y, 0)$*. We note that there is no flux through the sides of the cylinder as the normal at such points is orthogonal to the direction of the electric field. By Gauss' Law, we have that*

$$\begin{aligned}\frac{Q}{\varepsilon_0} = \frac{A\sigma}{\varepsilon_0} &= \oint_S \vec{E}\cdot d\vec{S}\\ &= 2AE(y)\end{aligned}$$

We thus see that $E(y) = \frac{\sigma}{2\varepsilon}$*. Hence*

$$\vec{E} = sgn(y)\frac{\sigma\vec{e}_y}{2\varepsilon_0}$$

Chapter 3

Vector and Scalar Potentials, Gauge Invariance

3.1 Vector potential

Theorem 3.1.1. *Let $\vec{B}$ be a vector field. Then the following four statements are equivalent:*

1. $\vec{\nabla} \cdot \vec{B} = 0$
2. $\int_S \vec{B} \cdot d\vec{S}$ *is independent of the choice of surface for a given fixed boundary*
3. $\oint_S \vec{B} \cdot d\vec{S} = 0$ *for any closed surface*
4. $\vec{B} = \vec{\nabla} \times \vec{A}$ *for some $\vec{A}$ called a* ***vector potential***

Proof. We shall prove the theorem in the order $(4) \implies (1) \implies (4), (1) \implies (3) \implies (1), (2) \implies (3) \implies (2)$

$\underline{(4) \implies (1)}$ Assume that $\vec{B} = \vec{\nabla} \times \vec{A}$ for some vector field $\vec{A}$. Then it follows automatically from the properties of vector fields that $\vec{\nabla} \cdot \vec{B} = 0$.

$(1) \implies (4)$ Assume that $\vec{\nabla} \cdot \vec{B} = 0$. We need to exhibit an $\vec{A}$ such that $\vec{B} = \vec{\nabla} \times \vec{A}$. Consider the following vector:

$$\begin{aligned} A_1 &= \int_0^z B_2(x, y, z')dz' + \int_0^y B_3(x, y', 0)dy' \\ A_2 &= -\int_0^z B_1(x, y, z')dz' \\ A_3 &= 0 \end{aligned}$$

We claim that $\vec{B} = \vec{\nabla} \times \vec{A}$. Indeed

$$\begin{aligned} \partial_y A_3 - \partial_z A_2 &= \partial_z \int_0^z B_1(x, y, z')dz' = B_1(x, y, z) \\ \partial_z A_1 - \partial_x A_3 &= \partial_z \int_0^z B_2(z, y, z')dz' = B_2(x, y, z) \\ \partial_x A_2 - \partial_y A_1 &= -\int_0^z (\partial_x B_1(x, y, z') + \partial_y B_2(x, y, z'))dz' - \partial_y \int_0^y B_3(x, y', 0)dy' \\ &= \int_0^z \partial_z B_3(x, y, z')dz' + B_3(x, y, 0) = B_3(x, y, z) \end{aligned}$$

where in the last equation we used the fact that $\vec{\nabla} \cdot \vec{B} = \partial_x B_1 + \partial_y B_2 + \partial_z B_3 = 0$.

$(1) \implies (3) \implies (1)$ By the divergence theorem, we have that

$$\oint_S \vec{B} \cdot d\vec{S} = \int_V \vec{\nabla} \cdot \vec{B} \, d^3x = 0$$

where the volume V is the one enclosed by S.

$(2) \implies (3) \implies (2)$ Consider two surfaces S_1 and S_2 with the same boundary. Then they must form a closed surface. Therefore we have that

$$\int_{S_1} \vec{B} \cdot d\vec{S} - \int_{S_2} \vec{B} \cdot d\vec{S} = \oint_{S_1 + S_2} \vec{B} \cdot d\vec{S} = 0$$

where the minus sign is because of the change of orientation of one of the surfaces.

□

3.2 Scalar Potential

Theorem 3.2.1. *Let $\vec{K}$ be a vector field. Then the following four statements are equivalent*

1. $\vec{\nabla} \times \vec{K} = 0$
2. $\int_\gamma \vec{K} \cdot d\vec{x}$ *is independent of the choice of path for a given fixed surface*
3. $\oint_\gamma \vec{K} \cdot d\vec{x}$ *for a closed path* γ
4. $\vec{K} = -\vec{\nabla}\phi$ *for some* ϕ *called a* ***scalar potential***

Proof. We shall prove the theorem only for (4) $\implies$ (1) $\implies$ (4). The rest follows in a similar case to the previous theorem.

(4) $\implies$ (1) Assume that $\vec{K} = -\vec{\nabla}\phi$ for some scalar function ϕ. We need to show that $\vec{\nabla} \times \vec{K} = 0$. We have that

$$\begin{aligned}[\vec{\nabla} \times (-\vec{\nabla}\phi)]_i &= -\epsilon_{ijk}\partial_j(\vec{\nabla}\phi)_j \\ &= -\epsilon_{ijk}\partial_j\partial_k\phi\end{aligned}$$

This is just 0 by the properties of the epsilon tensor and differential operators as shown earlier on in the notes.

(1) $\implies$ (4) Assume that $\vec{\nabla} \times \vec{K} = 0$. We need to exhibit a scalar function ϕ such that $\vec{K} = \vec{\nabla}\phi$. Consider the following function

$$\phi(\vec{x}) = -\int_\gamma \vec{K}(\vec{x}')\, dx'$$

where γ is a path between the origin and $\vec{x}$. We claim that $\vec{K} = \vec{\nabla}\phi$.
We first show that the definition of ϕ does not depend on the contour chosen. Consider two contours γ_1 and γ_2 from the origin to $\vec{x}$. We have that

$$\int_{\gamma_1} \vec{K}(\vec{x}') \cdot d\vec{x}' - \int_{\gamma_2} \vec{K}(\vec{x}') \cdot d\vec{x}' = \oint_{\gamma_1 - \gamma_2} \vec{K}(\vec{x}') \cdot d\vec{x}'$$

Now using Stoke's theorem, we see that

$$\oint_{\gamma_1-\gamma_2} \vec{K}(\vec{x}') \cdot d\vec{x}' = \int_{S_{\gamma_1-\gamma_2}} \left(\vec{\nabla} \times \vec{K}\right) \cdot d\vec{x} = 0$$

where we have used the fact that $\vec{\nabla} \times \vec{K} = 0$. It hence follows that ϕ is independent of parametrisation of the path.
We now calculuate the derivative component-wise:

$$\begin{aligned}
\partial_x \phi(\vec{x}) &= -\lim_{\epsilon\to 0} \frac{\phi(\vec{x}+\epsilon\vec{e}_x) - \phi(\vec{x})}{\epsilon} \\
&= -\lim_{\epsilon\to 0} \frac{1}{\epsilon} \left(\int_{\gamma_1} \vec{K}(\vec{x}') \cdot d\vec{x}' - \int_{\gamma_2} \vec{K}(\vec{x}') \cdot d\vec{x}' \right)
\end{aligned}$$

where γ_1 runs from the origin to $\vec{x}+\epsilon\vec{e}_x$ and γ_2 runs from the origin to $\vec{x}$. Combining these two contours, we get a path γ which runs from $\vec{x}$ to $\vec{x}+\epsilon\vec{e}_x$:

$$\begin{aligned}
\partial_x \phi(\vec{x}) &= -\lim_{\epsilon\to 0} \frac{1}{\epsilon} \left(\int_{\gamma_1} \vec{K}(\vec{x}') \cdot d\vec{x}' - \int_{\gamma_2} \vec{K}(\vec{x}') \cdot d\vec{x}' \right) \\
&= -\lim_{\epsilon\to 0} \frac{1}{\epsilon} \left(\int_{\gamma} \vec{K}(\vec{x}') \cdot d\vec{x}' \right) \\
&= -\lim_{\epsilon\to 0} \frac{1}{\epsilon} \left(\int_a^b \vec{K}(\gamma(s)) \frac{\partial \gamma(s)}{\partial s}\, ds \right)
\end{aligned}$$

Since the definition of ϕ does not depend on the shape of the contour, we can choose γ such that it is a straight line connecting the two points $\vec{x}$ and $\vec{x}+\epsilon\vec{e}_x$. Hence the derivative $\frac{\partial\gamma}{\partial s}$ is 1. Therefore

$$\begin{aligned}
\partial_x \phi(\vec{x}) &= -\lim_{\epsilon\to 0} \frac{1}{\epsilon} \left(\int_a^b \vec{K}(\gamma(s)) \frac{\partial \gamma(s)}{\partial s}\, ds \right) \\
&= -\lim_{\epsilon\to 0} \frac{1}{\epsilon} \left(\int_a^b \vec{K}(\gamma(s))\, ds \right) \\
&= -\lim_{\epsilon\to 0} \frac{1}{\epsilon} \left(\vec{K}(\vec{x}+\epsilon\vec{e}_x) - \vec{K}(\vec{x}) \right) \\
&= -\lim_{\epsilon\to 0} \frac{1}{\epsilon} \left(\vec{K}(\vec{x}) + \epsilon\vec{K}_x(\vec{x}) + o(|\epsilon^2|) - \vec{K}(\vec{x}) \right) \\
&= -\lim_{\epsilon\to 0} \frac{1}{\epsilon} \left(\epsilon\vec{K}_x(\vec{x}) + o(|\epsilon^2|) \right)
\end{aligned}$$

$$
\begin{aligned}
&= -\lim_{\epsilon\to 0}\left(\vec{K}_x(\vec{x}) + \frac{1}{\epsilon}o(|\epsilon^2|)\right) \\
&= -\lim_{\epsilon\to 0}\left(\vec{K}_x(\vec{x}) + o(|\epsilon|)\right) \\
&= -\vec{K}_x(\vec{x})
\end{aligned}
$$

We can apply the same steps to y and z to prove the claim.

□

3.3 Maxwell's Equations in terms of $\vec{A}$ and ϕ

Proposition 3.3.1. *Given a system of charges, Maxwell's first and fourth equations satisfy the following equation*

$$\vec{E} = -\vec{\nabla}\phi - \partial_t\vec{A}$$

for some scalar function ϕ and vector field $\vec{A}$.

Proof. M4 states that $\vec{\nabla}\cdot\vec{B} = 0$. By Section 3.1, we know that there exists a vector potential $\vec{A}$ such that $\vec{B} = \vec{\nabla}\times\vec{A}$.
Now, M1 states that $\vec{\nabla}\times\vec{E} = -\partial_t\vec{B}$. Combining this with the above, we have that

$$
\begin{aligned}
\vec{\nabla}\times\vec{E} &= -\partial_t\left(\vec{\nabla}\times\vec{A}\right) \\
&= -\vec{\nabla}\times\left(\partial_t\vec{A}\right)
\end{aligned}
$$

Rearranging we get that

$$
\begin{aligned}
\vec{\nabla}\times\vec{E} + \vec{\nabla}\times\left(\partial_t\vec{A}\right) &= 0 \\
\vec{\nabla}\times\left(E + \partial_t\vec{A}\right) &= 0
\end{aligned}
$$

By Section 3.2, it follows that

$$
\begin{aligned}
\vec{E} + \partial_t\vec{A} &= -\vec{\nabla}\phi \\
\vec{E} &= -\vec{\nabla}\phi - \partial_t\vec{A}
\end{aligned}
$$

□

Proposition 3.3.2. *Given a system of charges, Maxwell's third equation satisfies the following equation*

$$-\Delta\phi - \partial_t \vec{\nabla} \cdot \vec{A} = \frac{\rho}{\varepsilon_0}$$

Proof. Maxwell's third equation says that

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

By the previous proposition, we have that

$$\begin{aligned} \frac{\rho}{\varepsilon_0} &= \vec{\nabla} \cdot \vec{E} \\ &= \vec{\nabla} \cdot \left(-\vec{\nabla}\phi - \partial_t \vec{A}\right) \\ &= -\Delta\phi - \partial_t \vec{\nabla} \cdot \vec{A} \end{aligned}$$

□

Proposition 3.3.3. *Given a system of charges, Maxwell's second equation satisfies the following equation*

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\partial_t \phi\right) - \Delta\vec{A} + \frac{1}{c^2}\partial_t^2 \vec{A} = \frac{\vec{j}}{c^2 \varepsilon_0}$$

Proof. Maxwell's second equation says that

$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2}\partial_t \vec{E} + \frac{1}{c^2 \varepsilon_0}\vec{j}$$

We see that

$$\begin{aligned} \frac{\vec{j}}{c^2\varepsilon_0} &= \vec{\nabla} \times \vec{B} - \frac{1}{c^2}\partial_t \vec{E} \\ &= \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) - \frac{1}{c^2}\partial_t \left(-\vec{\nabla}\phi - \frac{1}{c^2}\partial_t \vec{A}\right) \\ &= \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) + \frac{1}{c^2}\vec{\nabla}\partial_t \phi + \frac{1}{c^2}\partial_t^2 \vec{A} \\ &= \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \Delta\vec{A} + \frac{1}{c^2}\vec{\nabla}\partial_t \phi + \frac{1}{c^2}\partial_t^2 \vec{A} \\ &= \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\partial_t \phi\right) - \Delta\vec{A} + \frac{1}{c^2}\partial_t^2 \vec{A} \end{aligned}$$

where we have used the following relation

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{K}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{K}\right) - \Delta \vec{K}$$

□

3.4 Gauge Transformations and Gauge Invariance

The transformations from the potentials $\vec{E}, \vec{B}$ to the fields $\vec{A}, \phi$ are not unique. There could be multiple potentials which give rise to the same fields. They are physically indistinguishable snce only $\vec{E}$ and $\vec{B}$ are measurable quantities.
Assume that two vector potentials $\vec{A} \neq \vec{A}'$ result in the same magnetic field $\vec{B}$. We have that $B = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{A}'$. It follows that $0 = \vec{\nabla} \times (\vec{A} - \vec{A}')$. Then Theorem 3.2.1 implies that $\vec{A} - \vec{A}' = -\vec{\nabla}\Lambda$ for some scalar function Λ. After rearranging this equation, it follows that

$$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda \tag{3.1}$$

Hence any transformation of $\vec{A}$ that leaves $\vec{B}$ invariant must involve only the gradient of some scalar function. However, this transformation may not necessarily leave $\vec{E}$ invariant. In order to achieve this, we assume the following

$$\vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A} = -\vec{\nabla}\phi' - \partial_t \vec{A}'$$

for some scalar functions $\phi \neq \phi'$ and vector fields $\vec{A} \neq \vec{A}'$. By the above analysis, we know that $\vec{A}'$ must differ from $\vec{A}$ by the gradient of a scalar function Λ. Therefore

$$\begin{aligned}
& -\vec{\nabla}\phi - \partial_t \vec{A} = -\vec{\nabla}\phi' - \partial_t\left(\vec{A} + \vec{\nabla}\Lambda\right) \\
\iff \ & \vec{\nabla}\phi' - \vec{\nabla}\phi = \partial_t \vec{A} - \partial_t\left(\vec{A} + \vec{\nabla}\Lambda\right) \\
\iff \ & \vec{\nabla}\phi' - \vec{\nabla}\phi = -\partial_t \vec{\nabla}\Lambda \\
\iff \ & \vec{\nabla}\left(\phi' - \phi\right) = \vec{\nabla}\left(-\partial_t \Lambda\right) \\
\iff \ & \phi' - \phi = -\partial_t \Lambda
\end{aligned} \tag{3.2}$$

Definition 3.4.1. *Let $\vec{A}$ be a vector potential for the magnetic field $\vec{B}$ and ϕ a scalar potential for the electric field. Consider a scalar function Λ. Then the transformations*

$$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda$$
$$\phi' = \phi - \partial_t\Lambda$$

are called ***gauge transformations****. They give rise to a vector potential $\vec{A}'$ and scalar potential ϕ' that leave $\vec{B}$ and $\vec{E}$ invariant.*

3.5 Lorentz Gauge

We can reduce the complexity of the equations found in Section 3.3 using the information we deduced in Section 3.2. In particular, if we can find gauge transformations that satisfy the **Lorentz Gauge** condition

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\partial_t\phi = 0$$

then the equations in Section 3.3 can be reduced to simpler forms.

Theorem 3.5.1. *Let $\vec{A}$ be a vector field and ϕ a scalar field. Then we can always find a scalar field Λ such that the Lorentz Gauge condition*

$$\vec{\nabla} \cdot \vec{A}' + \frac{1}{c^2}\partial_t\phi' = 0$$

is satisfied where $\vec{A}' = \vec{A} + \vec{\nabla}\Lambda$ and $\phi' = \phi - \partial_t\Lambda$

Proof. If $\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\partial_t\phi = 0$ then we can choose $\Lambda = 0$ and we are done. Hence assume that $\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\partial_t\phi = \psi$ for some non-zero ψ. Then

$$\psi = \vec{\nabla} \cdot \left(\vec{A}' - \vec{\nabla}\Lambda\right) + \frac{1}{c^2}\partial_t(\phi' + \partial_t\Lambda)$$
$$\implies \vec{\nabla}' + \frac{1}{c^2}\partial_t\phi' = \psi + \Delta\Lambda - \frac{1}{c^2}\partial_t^2\Lambda$$

It is known that the equation

$$-\Delta\Lambda + \frac{1}{c^2}\partial_t^2\Lambda = \psi$$

can always be solved for Λ. Hence we can always find a scalar Λ such that the Lorentz Gauge condition is satisfied. $\square$

In light of the previous theorem, we can always assume that the Lorentz Gauge condition is satisfied. This allows us to simplify the equations into the following forms:

$$-\Delta\vec{A} + \frac{1}{c^2}\partial_t^2\vec{A} = \frac{\vec{j}}{c^2\varepsilon_0}$$
$$-\Delta\phi + \frac{1}{c^2}\partial_t^2\phi = \frac{\rho}{\varepsilon_0}$$
$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\partial_t\phi = 0$$

Chapter 4

Distributions and Generalized Functions

Definition 4.0.2. *We define the set S to be the linear space of smooth functions satisfying*

$$S = \left\{ f(\vec{x}) \;\middle|\; |f(\vec{x})| < \frac{c_n}{|\vec{x}|^n},\ c_n \in \mathbb{R}, n \in \mathbb{N} \right\}$$

Definition 4.0.3. *We define the* ***norm*** *on S to be the function*

$$||f - g|| = \sqrt{\int |f - g|^2 \, d^n x}$$

where f and g are arbitrary functions.

Definition 4.0.4. *We define the* ***distributions*** *on S to be the linear functionals on S. They map any function in S to a real number. We will usually deal with the following type of distribution*

$$D_f[g] = \int f(x)g(x)d^n x$$

Definition 4.0.5. *We say that a distribution $D[\cdot]$ is* ***continuous*** *if for any sequence of probe functions $(g_n)_{n\in\mathbb{N}}$ that is convergent with respect to the norm on S, the sequence of real numbers*

$$c_n = D[g_n]$$

is also convergent.

Definition 4.0.6. *The* ***delta function*** $\delta(x-a)$ *is defined in terms of the linear functional* $D_{\delta(x-a)}$ *satisfying the following*

$$D_{\delta(x-a)} = g(a)$$

Proposition 4.0.7. *Consider a distribution* $D_f[g]$*. Then the derivative of the distribution* $D_{\partial_i f}[g]$ *equals* $-D[\partial_i g]$*. In the general case we define the derivative of a distribution to be* $\partial_i D[g] = -D[\partial_i g]$*.*

Proof. We have that

$$\begin{aligned}
D_{\partial_i f}[g] &= \int_{\mathbb{R}} (\partial_i f) g \; dx \\
&= [fg]_{-\infty}^{\infty} - \int_{\mathbb{R}} f \partial_i g \; dx \\
&= - \int_{\mathbb{R}} f \partial_i g \; dx \\
&= -D_f[\partial_i g]
\end{aligned}$$

where we have integrated by parts and used the fact that f and g vanish sufficiently fast enough at infinity. □

Example 4.0.8. *Let* $a \in \mathbb{R}$ *and consider the function*

$$\theta_a(x) = \begin{cases} 1 & \text{if } x \geq a \\ 0 & \text{if } x < a \end{cases}$$

$\theta_a(x)$ *is clearly not differentiable or even continuous. However, we can define its derivative in terms of distributions.*
The distribution corresponding to $\theta_a(x)$ *is*

$$D_{\theta_a}[g] = \int_{-\infty}^{\infty} \theta_a(x) g(x) \; dx = \int_a^{\infty} g(x) \; dx \tag{4.1}$$

According to the definition of the derivative of a distribution, we have that

$$D_{\partial_x \theta_a}[g] = -D_{\theta_a}[\partial_x g]$$

inserting this into Equation (4.1), we get

$$\begin{aligned}
D_{\partial_x \theta_a}[g] &= - \int_a^{\infty} \partial_x g(x) \; dx \\
&= -g(\infty) + g(a) = g(a)
\end{aligned}$$

where we have used the fundamental theorem of calculus and the fact that the probe function $g(x)$ *decays at infinity. We can now see that* $D_{\partial_x \theta_a} = D_{\delta(x-a)}$*.*

4.1 Green's functions

Definition 4.1.1. *Let $\mathcal{D}$ be some differential operator. Then we define the* ***Green function*** *$G(\vec{x}, \vec{a})$ for $\mathcal{D}$ to be the one that satisfies the following*

$$\mathcal{D}G(\vec{x}, \vec{a}) = \delta(\vec{x} - \vec{a})$$

Example 4.1.2. *For the Laplace operator Δ, we have that*

$$G(\vec{x}, \vec{y}) = -\frac{1}{4\pi} \frac{1}{|\vec{x} - \vec{y}|}$$

Theorem 4.1.3. *Let $\mathcal{D}$ be a differential operator and $G(\vec{x}, \vec{a})$ its Green function. Consider two functions f and g. Then the solution to the inhomogeneous equation $\mathcal{D}f = g$ is given by*

$$f(\vec{x}) = \int_{\mathbb{R}^n} G(\vec{x}, \vec{a}) g(\vec{a}) \, d^n a$$

Proof. We have that

$$\begin{aligned} \mathcal{D}f(\vec{x}) &= \int_{\mathbb{R}^n} \mathcal{D}G(\vec{x}, \vec{a}) g(\vec{a}) \, d^n a \\ &= \int_{\mathbb{R}^n} \delta(\vec{x} - \vec{a}) g(\vec{a}) \, d^n a \\ &= g(\vec{x}) \end{aligned}$$

□

Corollary 4.1.4. *Consider the Laplace operator Δ. Then the solution for $\Delta f = g$ is given by*

$$f = -\int \frac{g(x')}{4\pi|\vec{x} - \vec{x}'|} \, d^3 x'$$

Proof. It suffices to show that the Green function for the Laplace operator is $-\frac{1}{4\pi|\vec{x}-\vec{y}|}$. The corollary then follows from the previous theorem.
The function has a singularity at the origin hence we can only define its derivative in terms of distributions. Without loss of generality, we can set $\vec{y} = 0$. We need to show that

$$\begin{aligned} \partial_i \partial_i D_{-\frac{1}{4\pi|\vec{x}|}}[g(\vec{x})] &= D_{-\frac{1}{4\pi|\vec{x}|}}[\partial_i \partial_i g(\vec{x})] \\ &= D_{\delta(\vec{x})}[g(\vec{x})] \\ &= g(\vec{0}) \end{aligned}$$

We have that

$$D_{-\frac{1}{4\pi|\vec{x}|}}[\partial_i\partial_i g(\vec{x})] = \int_{\mathbb{R}^3}\left(-\frac{1}{4\pi|\vec{x}|}\right)\partial_i\partial_i g(\vec{x})\ d^3x$$

We thus have to show that the integral is equal to $g(\vec{0})$. Since the integral is divergent at the origin, we can split $\mathbb{R}^3$ into a sphere of radius ϵ denoted V_ϵ and the complement $\mathbb{R}^3\backslash V_\epsilon$. We hence have that

$$\begin{aligned}\int_{\mathbb{R}^3}\left(-\frac{1}{4\pi|\vec{x}|}\right)\partial_i\partial_i g(\vec{x})\ d^3x &= \underbrace{\int_{V_\epsilon}\left(-\frac{1}{4\pi|\vec{x}|}\right)\partial_i\partial_i g(\vec{x})\ d^3x}_{a} \\ &\quad + \underbrace{\int_{\mathbb{R}^3\backslash V_\epsilon}\left(-\frac{1}{4\pi|\vec{x}|}\right)\partial_i\partial_i g(\vec{x})\ d^3x}_{b}\end{aligned}$$

We show that a vanishes as $\epsilon \to 0$. Since $\partial_i\partial_i g(\vec{x})$ is a continuous function, we are, by the mean value theorem, guaranteed the existence of an $\vec{x}_0 \in \mathbb{R}^3$ such that

$$a = -\partial_i\partial_j g(\vec{x})\Big|_{\vec{x}_0}\int_{V_\epsilon}\frac{1}{4\pi|\vec{x}|}\ d^3x$$

Setting $\vec{c} = -\partial_i\partial_j g(\vec{x})\Big|_{\vec{x}_0}$ we see that

$$\begin{aligned}\lim_{\epsilon\to 0} a &= \lim_{\epsilon\to 0}\left(\vec{c}\int_{V_\epsilon}\frac{1}{4\pi|\vec{x}|}\ d^3x\right) \\ &= -\vec{c}\lim_{\epsilon\to 0}\int_0^\epsilon\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\frac{r^2 sin(\phi)}{4\pi r}\ d\phi d\theta dr \\ &= -\vec{c}\lim_{\epsilon\to 0}\int_0^\epsilon\frac{r}{4\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} sin(\phi)\ d\phi d\theta dr \\ &= -\vec{c}\lim_{\epsilon\to 0}\int_0^\epsilon\frac{r}{4\pi}\int_0^{2\pi}\big[-cos(\phi)\big]_0^{\frac{\pi}{2}} d\theta dr \\ &= -\vec{c}\lim_{\epsilon\to 0}\int_0^\epsilon\frac{r}{4\pi}\int_0^{2\pi} d\theta dr \\ &= -\vec{c}\lim_{\epsilon\to 0}\int_0^\epsilon\frac{r}{2}\ dr \\ &= -\vec{c}\lim_{\epsilon\to 0}\frac{\epsilon^2}{4}\ dr \\ &= \vec{0}\end{aligned}$$

Hence we only need to consider b. We first derive the integration by parts analogue for volume integrals. By the divergence theorem, we have that

$$\int_V \vec{\nabla} \cdot \vec{B} \, d^3x = \oint_{\partial V} \vec{B} \, d\left(\partial \vec{V}\right)$$

Letting $\vec{B} = f\vec{h}$, it follows that

$$\int_V \vec{\nabla} \cdot \left(f\vec{h}\right) \, d^3x = \oint_{\partial V} f\vec{h} \, d\left(\partial \vec{V}\right)$$

By the product rule we have

$$\int_V f\left(\vec{\nabla} \cdot \vec{h}\right) \, d^3x + \int_V \vec{\nabla} f \cdot \vec{h} \, d^3x = \oint_{\partial V} f\vec{h} \, d\left(\partial \vec{V}\right) \tag{4.2}$$

We can therefore rearrange Equation (4.2) to obtain the desired integration by parts rule.
Consider again b. By integration by parts, we have that

$$b = \int_{\mathbb{R}^3 \setminus V_\epsilon} \underbrace{\left(-\frac{1}{4\pi|\vec{x}|}\right)}_{f(\vec{x})} \partial_i \underbrace{\partial_i g(\vec{x})}_{h_i(\vec{x})} = -\int_{\mathbb{R}^3 \setminus V_\epsilon} \partial_i \left(-\frac{1}{4\pi|\vec{x}|}\right) \partial_i g(\vec{x}) \, d^3x$$
$$+ \oint_{S_{\mathbb{R}^3 \setminus V_\epsilon}} \left(-\frac{1}{4\pi|\vec{x}|}\right) \partial_i g(\vec{x}) \, dS_i$$

Again applying integration by parts, we can see that

$$b = -\int_{\mathbb{R}^3 \setminus V_\epsilon} \underbrace{\partial_i \left(-\frac{1}{4\pi|\vec{x}|}\right)}_{h_i(\vec{x})} \partial_i \underbrace{g(\vec{x})}_{f(\vec{x})} \, d^3x + \oint_{S_{\mathbb{R}^3 \setminus V_\epsilon}} \left(-\frac{1}{4\pi|\vec{x}|}\right) \partial_i g(\vec{x}) \, dS_i$$
$$= \int_{\mathbb{R}^3 \setminus V_\epsilon} \partial_i \partial_i \left(-\frac{1}{4\pi|\vec{x}|}\right) g(\vec{x}) \, d^3x + \underbrace{\oint_{S_{\mathbb{R}^3 \setminus V_\epsilon}} \left(-\frac{1}{4\pi|\vec{x}|}\right) \partial_i g(\vec{x}) \, dS_i}_{c}$$
$$- \underbrace{\oint_{S_{\mathbb{R}^3 \setminus V_\epsilon}} \partial_i \left(-\frac{1}{4\pi|\vec{x}|}\right) g(\vec{x}) \, dS_i}_{d}$$

To show that the first term is zero, we use the following identity:

$$\begin{aligned}
\partial_i \partial_i \frac{1}{|\vec{x}|} &= \partial_i \partial_i \frac{1}{r} \\
&= \partial_i \left(\frac{-x_i}{r^3} \right) \\
&= -\frac{3}{r^3} + 3\frac{x_i x_i}{r^5} \\
&= 0
\end{aligned}$$

We are thus left with two surface integrals c and d. Let us first consider d. Since this is a surface integral over a sphere of radius ϵ, we can replace all occurences of $r = |\vec{x}|$ with ϵ. Using $\partial_i \left(-\frac{1}{4\pi\vec{x}}\right) = \frac{1}{4\pi}\frac{x_i}{r^3}$ we see that

$$d = \oint_{S_{\mathbb{R}^3 \setminus V_\epsilon}} \frac{x_i}{4\pi\epsilon^3} g(\vec{x})\, dS_i$$

Since we must choose the direction of the normal vector to point inside of the sphere, we take $d\vec{S} = \vec{n} dS = -\frac{\vec{x}}{r} dS$. This allows us to replace dS_i by $-\frac{x_i}{r} = -\frac{x_i}{\epsilon}$:

$$\begin{aligned}
d &= -\oint_{S_{\mathbb{R}^3 \setminus V_\epsilon}} \frac{x_i}{4\pi\epsilon^3} g(\vec{x}) \frac{x_i}{\epsilon}\, dS \\
&= -\oint_{S_{\mathbb{R}^3 \setminus V_\epsilon}} \frac{x_i x_i}{4\pi\epsilon^4} g(\vec{x})\, dS \\
&= -\oint_{S_{\mathbb{R}^3 \setminus V_\epsilon}} \frac{r^2}{4\pi\epsilon^4} g(\vec{x})\, dS \\
&= -\frac{1}{4\pi\epsilon^2} \oint_{S_{\mathbb{R}^3 \setminus V_\epsilon}} g(\vec{x})\, dS
\end{aligned}$$

By the mean value theorem, we are guaranteed the existence of an $\vec{x}_0 \in \mathbb{R}^3$

such that

$$\begin{aligned}
d &= -\frac{1}{4\pi\epsilon^2} \oint_{S_{\mathbb{R}^3 \setminus V_\epsilon}} g(\vec{x})\, dS \\
&= -\frac{g(\vec{x}_0)}{4\pi\epsilon^2} \oint_{S_{\mathbb{R}^3 \setminus V_\epsilon}} dS
\end{aligned}$$

Now, the surface area of the sphere is $4\pi\epsilon^2$, hence we have that

$$\begin{aligned}
d &= -\frac{g(\vec{x}_0)}{4\pi\epsilon^2} \oint_{S_{\mathbb{R}^3 \setminus V_\epsilon}} dS \\
&= -g(x_0)
\end{aligned}$$

In the limit $\lim_{\epsilon \to 0}$, $\vec{x}_0 \to \vec{0}$. Hence $d = -g(\vec{0})$. We only need to show that c vanishes. Indeed, replacing r with ϵ and using the mean value theorem, we have that

$$\begin{aligned}
\lim_{\epsilon \to 0} c &= \lim_{\epsilon \to 0} \oint_{S_{\mathbb{R}^3 \setminus V_\epsilon}} \left(-\frac{1}{4\pi|\vec{x}|} \right) \partial_i g(\vec{x})\, dS_i \\
&= \lim_{\epsilon \to 0} \oint_{S_{\mathbb{R}^3 \setminus V_\epsilon}} \left(-\frac{1}{4\pi\epsilon} \right) \partial_i g(\vec{x}) n_i\, dS \\
&= \lim_{\epsilon \to 0} \left(\frac{-1}{4\pi\epsilon} \right) \partial_i g(x)\big|_{\vec{x}_0} n_i \oint_{S_{\mathbb{R}^3 \setminus V_\epsilon}} dS \\
&= \lim_{\epsilon \to 0} -\epsilon \partial_i g(x)\big|_{\vec{x}_0} n_i \\
&= 0
\end{aligned}$$

□

4.2 General Solution to the Poisson equation

In the previous section, we derived the following solution to the Poisson equation $\Delta f = g$:

$$f(\vec{x}) = -\frac{1}{4\pi} \int \frac{g(\vec{y})}{|\vec{x} - \vec{y}|}\, d^3x$$

Lemma 4.2.1. *Let $g(\vec{x}) = 0$. Then the only decaying at infinity solution to the Poisson equation is $f(\vec{x}) = 0$.*

Proof. Assume that there exists a non-zero decaying at infinity function $H(\vec{x})$ such that $\Delta H(\vec{x}) = 0$. Then the integral $\int \partial_i H \partial_i H$ must be strictly positive. By integration by parts, we have that

$$\begin{aligned} 0 < \int_{\mathbb{R}^3} \partial_i H \partial_i H \; d^3x &= -\int_{\mathbb{R}^3} H \partial_i \partial_i \; d^3x \qquad = -\int_{\mathbb{R}^3} H \partial_i \partial_i H \\ &= -\int_{\mathbb{R}^3} H \Delta H \\ &= 0 \end{aligned}$$

We therefore obtain a contradiction and we must have that $H(\vec{x}) = 0$. □

Theorem 4.2.2. *The solution of the Poisson equation is unique in the class of decaying at infinity functions.*

Proof. Assume that the solution is not uniquely. Then there exist distinct functions f and f' such that $\Delta f = g$ and $\Delta f' = g$. Due to linearity of the Laplace operator, we have that $\Delta(f - f') = \Delta f - \delta f' = g - g = 0$. By the above lemma, we have that $f - f' = 0 \implies f = f'$ which is a contradiction. Hence the solution is unique. □

Remark. *There are some non-trivial solutions to the Poisson equation that grow at infinity. For example, $f(\vec{x}) = a + b_i x_i$.*

Chapter 5

Static fields

We now consider the case where ρ and $\vec{j}$ are time-independent or **static**. The charges can move around the space but ρ and $\vec{v}$ are constant at each point. We therefore have that $\partial_t \vec{E} = 0$ and $\partial_t \vec{B} = 0$. This simplifies the Maxwell equations into the following forms:

$$\vec{\nabla} \times \vec{E} = 0$$
$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$$
$$\vec{\nabla} \cdot \vec{B} = 0$$

where $\mu_0 = \frac{1}{c^2 \varepsilon_0}$ is the **vacuum permeability of free space** constant. The relations to the vector and scalar potentials are also much simpler:

$$-\Delta \vec{A} = \mu_0 \vec{j}$$
$$-\Delta \phi = \frac{\rho}{\varepsilon_0}$$
$$\vec{\nabla} \cdot \vec{A} = 0$$

The first two of these equations are Poisson equations and have solutions

$$\phi = \frac{1}{4\pi\varepsilon_0} \int_{\mathbb{R}^3} \frac{\rho(\vec{y})}{|\vec{x} - \vec{y}|} \, d^3y$$
$$\vec{A} = \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \frac{\vec{j}(\vec{y})}{|\vec{x} - \vec{y}|} \, d^3y$$

which give the following expressions for the electric and magnetic fields:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_{\mathbb{R}^3} \frac{\rho(\vec{y})(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3}\, d^3y$$
$$\vec{B} = \frac{\mu_0}{4\pi}\int_{\mathbb{R}^3} \frac{\vec{j}(\vec{y})\times(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3}\, d^3y$$

5.1 Point-like charge

Example 5.1.1. *Consider a point-like charge situated at $\vec{x}_1$ with charge density given by $\rho(\vec{x}) = q\delta(\vec{x}-\vec{x}_1)$. We can calculate the electric potential as follows:*

$$\phi = \frac{1}{4\pi\varepsilon_0}\int_{\mathbb{R}^3} \delta(\vec{y}-\vec{x_1})\frac{q}{|\vec{x}-\vec{y}|}\, d^3y$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{x}-\vec{x}_1|}$$

The electric field is then given by

$$\vec{E} = -\vec{\nabla}\phi$$
$$= -\frac{1}{4\pi\varepsilon_0}\vec{\nabla}\frac{q}{|\vec{x}-\vec{x}_1|}$$
$$= \frac{q}{4\pi\varepsilon_0}\frac{\vec{x}-\vec{x}_1}{|\vec{x}-\vec{x}_1|^3}$$

5.2 Superposition Principle

Theorem 5.2.1. *(Superposition Principle)*
The electric and magnetic fields created by a combination of charges and currents is equal to the sum of the fields created by each of the charges individually.

Proof. This follows from linearity of the Laplace operator. □

5.3 Electric Field of a Charged Infinite Straight Line

Example 5.3.1. *Consider a charged infinite straight line. We can characterise the charge density of a section of length h by a linear density λ such that $q = \lambda h$. Dividing each cut into small sections of length dz, we can treat each length as point-like charges. If we take the limit of $dz \to 0$, we arrive at the following integral:*

$$\phi = \int_{-H_1}^{H_2} \frac{1}{4\pi\varepsilon_0} \frac{\lambda dz}{\sqrt{x^2 + y^2 + z^2}}$$

where we consider a finite line going from $-H_1$ to H_2 along the z axis. Solving the integral, we get

$$\phi = \frac{\lambda}{4\pi\varepsilon_0} \log \left(\frac{\sqrt{1 + \frac{H_2^2}{l^2}} + \frac{H_2}{l}}{\sqrt{1 + \frac{H_1^2}{l^2}} - \frac{H_1}{l}} \right)$$

where $l = \sqrt{x^2 + y^2}$. Now it follows that

$$\begin{aligned}
\phi &= \frac{\lambda}{4\pi\varepsilon_0} \log \left(\frac{\sqrt{1 + \frac{H_2^2}{l^2}} + \frac{H_2}{l}}{\sqrt{1 + \frac{H_1^2}{l^2}} - \frac{H_1}{l}} \times \frac{\sqrt{1 + \frac{H_1^2}{l^2}} + \frac{H_1}{l}}{\sqrt{1 + \frac{H_1^2}{l^2}} + \frac{H_1}{l}} \right) \\
&= \frac{\lambda}{4\pi\varepsilon_0} \log \left(\frac{\left(\sqrt{1 + \frac{H_2^2}{l^2}} + \frac{H_2}{l} \right) \left(\sqrt{1 + \frac{H_1^2}{l^2}} + \frac{H_1}{l} \right)}{1 + \frac{H_1^2}{l^2} - \frac{H_1^2}{l^2}} \right) \\
&= \frac{\lambda}{4\pi\varepsilon_0} \log \left[\left(\sqrt{1 + \frac{H_2^2}{l^2}} + \frac{H_2}{l} \right) \left(\sqrt{1 + \frac{H_1^2}{l^2}} + \frac{H_1}{l} \right) \right]
\end{aligned}$$

Since H_1 and H_2 are very large, we can approximate the square roots by removing the ones:

$$\phi = \frac{\lambda}{4\pi\varepsilon_0}\log\left[\left(\sqrt{\frac{H_2^2}{l^2}}+\frac{H_2}{l}\right)\left(\sqrt{\frac{H_1^2}{l^2}}+\frac{H_1}{l}\right)\right]$$
$$\phi = \frac{\lambda}{4\pi\varepsilon_0}\log\left[\left(\frac{2H_2}{l}\right)\left(\frac{2H_1}{l}\right)\right]$$
$$\phi = \frac{\lambda}{4\pi\varepsilon_0}\log\frac{4H_2H_1}{l^2}$$
$$\phi = -\frac{\lambda}{4\pi\varepsilon_0}\log(x^2+y^2)+c$$

for some constant c. Therefore, the electric field is

$$\begin{aligned}\vec{E} &= -\vec{\nabla}\phi \\ &= \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{x^2+y^2}\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}\end{aligned}$$

5.4 Integral form of Maxwell's Equations

For the two Maxwell's Equations $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$ and $\vec{\nabla}\cdot\vec{B} = 0$, we can integrate over a volume V and apply the divergence theorem:

$$\int_V \vec{\nabla}\cdot\vec{E}\, d^3x = \oint_{S_V}\vec{E}\cdot d\vec{S} = \int_V \frac{\rho}{\varepsilon_0}\, d^3x = \frac{Q_V}{\varepsilon_0}$$
$$\int_V \vec{\nabla}\cdot\vec{B}\, d^3x = \oint_{S_V}\vec{B}\cdot d\vec{S} = \int_V 0\, d^3x = 0$$

For the two Maxwell's Equations $\vec{\nabla}\times\vec{E} = -\partial_t\vec{B}$ and $\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\partial_t\vec{E}+\frac{1}{c^2\varepsilon_0}\vec{j}$, we can integrate over a surface S and apply Stoke's theorem:

$$\int_S \vec{\nabla}\times\vec{E}\cdot d\vec{S} = \oint_{\partial S}\vec{E}\cdot d\vec{x} = \int_S -\partial_t\vec{B}\cdot d\vec{S}$$
$$\int_S \vec{\nabla}\times\vec{B}\cdot d\vec{S} = \oint_{\partial S}\vec{B}\cdot d\vec{x} = \int_S \frac{1}{c^2}\partial_t\vec{E}+\frac{1}{c^2\varepsilon_0}\vec{j}\cdot d\vec{S}$$

With regards to magnetostatics, the last equation is useful when we drop the $\partial_t\vec{E}$ term. It is known as **Ampere's circuital law**.

5.5 Magnetic field of a long wire

We first note that the flux of the magnetic field is 0 for any closed surface. This follows from the integral form of Maxwell's equations. If we take a cylinder surrounding a section of the wire, we see that $\vec{B}$ should be tangent to the cylinder. We now use Ampere's circuital law to compute the length of the magnetic field, using a circle of radius r as the integration contour γ. Ampere's law states that

$$\oint_{\partial S} \vec{B} \cdot d\vec{x} = \int_S \frac{1}{c^2 \varepsilon_0} \vec{j} \cdot d\vec{S}$$

Now since the magnetic field is parallel to the direction of the contour γ, we have that

$$\begin{aligned}\oint_\gamma \vec{B} \cdot d\vec{x} &= \oint_\gamma |\vec{B}||d\vec{x}|cos(0) \\ &= |\vec{B}| \oint_\gamma dx \\ &= |\vec{B}| \times 2\pi r\end{aligned}$$

Now since $\vec{j} = \vec{e}_z I \delta(x)\delta(y)$ for some constant I, we see that

$$\begin{aligned}\int_S \frac{1}{c^2 \varepsilon_0} \vec{j} \cdot d\vec{S} &= \mu_0 I \int_S \vec{e}_z \delta(x)\delta(y) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \ dxdy \\ &= \mu_0 I \int_S \delta(x)\delta(y) \ dxdy \\ &= \mu_0 I \\ &\implies B(r) = \frac{\mu_0 I}{2\pi r}\end{aligned}$$

We could also use the general solution for $\vec{A}$ to solve the problem. Again using $\vec{j} = \vec{e}_z I \delta(x)\delta(y)$, we have that

$$\begin{aligned}\vec{A} &= \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \frac{\vec{j}(\vec{x}')}{|\vec{r} - \vec{r}'|} \ d^3x' \\ &= \frac{\vec{e}_z I \mu_0}{4\pi} \int \frac{\delta(x')\delta(y')dx'dy'dz'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}\end{aligned}$$

$$= \frac{\vec{e}_z I \mu_0}{4\pi} \int \frac{1}{\sqrt{x^2 + y^2 + (z - z')^2}} dz'$$
$$= \frac{\vec{e}_z I \mu_0}{4\pi} log(x^2 + y^2) + c$$

This implies that the magnetic field is

$$\vec{B} = \frac{\vec{e}_z I \mu_0}{2\pi r} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

5.6 Dipole Field, Multipole Expansion

Definition 5.6.1. *Consider a system of charges consisting of two positive and negative charges with an overall charge of zero. This system is referred to as a **dipole**.*

Remark. *Even though the overall charge is vanishing, an electric field is still created. Such an electric field results in interaction between dipoles close to each other.*

Lemma 5.6.2. *Consider the function $f(\vec{x} - \vec{x'}) = \frac{1}{|\vec{x} - \vec{x}'|}$. Then its Taylor expansion is given by*

$$\frac{1}{|\vec{x} - \vec{x}'|} = \frac{1}{|\vec{x}|} + \frac{\vec{x} \cdot \vec{y}}{|\vec{x}|^3} + \frac{3(\vec{x} \cdot \vec{y})^2 - |\vec{x}|^2 |\vec{y}|^2}{2|\vec{x}|^5} + \dots$$

Proof. In general, the Taylor series for a function of the form $f(\vec{x} + \vec{a})$ is given by

$$f(\vec{x} + \vec{a}) = f(\vec{x}) + \partial_i f(\vec{x}) a_i + \frac{1}{2} \partial_i \partial_j f(\vec{x}) a_i a_j + \dots$$

Now, we have that

$$f(\vec{x}) = \frac{1}{|\vec{x} - \vec{x}'|}$$
$$\delta_i f(\vec{x}) = -\frac{x_i}{|\vec{x}|^3}$$
$$\delta_i \delta_j = -\frac{\delta_i j}{|\vec{x}|^3} + 3\frac{x_i x_j}{|\vec{x}|^5}$$

Inserting this into the general formula for the Taylor series, we arrive at the desired result. □

Proposition 5.6.3. *Consider two point like charges with zero total charge seperated by a small amount $\vec{d}$ centered at the origin. Then the potential for this system is given by*

$$\phi = \frac{q\vec{x} \cdot \vec{d}}{4\pi\varepsilon_0 |\vec{x}|^3}$$

Proof. The potential for this system is given by the sum of the potentials of the two charges:

$$\begin{aligned}
\phi &= \frac{q}{4\pi\varepsilon_0 \left|\vec{x} - \frac{\vec{d}}{2}\right|} - \frac{q}{4\pi\varepsilon_0 \left|\vec{x} - \frac{\vec{d}}{2}\right|} \\
&= \frac{q}{4\pi\varepsilon_0} \left(\frac{1}{\left|\vec{x} - \frac{\vec{d}}{2}\right|} - \frac{1}{\left|\vec{x} - \frac{\vec{d}}{2}\right|} \right) \\
&= \frac{q}{4\pi\varepsilon_0} \left(f(\vec{x}) + \frac{\partial_i f(\vec{x})\vec{d}}{2} - f(\vec{x}) + \frac{\partial_i f(\vec{x})\vec{d}}{2} + \dots \right) \\
&= \frac{q}{4\pi\varepsilon_0} \left(-\frac{\vec{x}\vec{d}}{2|\vec{x}|^3} - \frac{\vec{x}\vec{d}}{2|\vec{x}|^3} + \dots \right)
\end{aligned}$$

In the limit $\vec{d} \to \vec{0}$ we get a very good approximation and

$$\phi = \frac{q\vec{x}\vec{d}}{4\pi\varepsilon_0 |\vec{x}|^3}$$

□

Remark. *The electric field is given by*

$$\begin{aligned}
\vec{E}_i &= -\partial_i \phi \\
&= \partial_i \left(\frac{qx_j d_j}{4\pi\varepsilon_0 r^3} \right) \\
&= \frac{q\delta_i j d_j}{4\pi\varepsilon_0 r^3} - 3\frac{qx_i x_j d_j}{4\pi\varepsilon_0 r^5} \\
&= \frac{qr^2 d_i - 3x_i(\vec{j} \cdot \vec{d})}{4\pi\varepsilon_0 r^5}
\end{aligned}$$

Definition 5.6.4. *Let q be the magnitude of the charges of two point-like charges in a dipole system and $\vec{d}$ the distance between them. Then we define the **dipole moment** to be the quantity $\vec{p} = q\vec{d}$.*

Remark. *With the dipole moment, we can reformulate the definition of the electric field as*

$$\vec{E} = \frac{r^2\vec{p} - 3\vec{x}(\vec{x}\cdot\vec{p})}{4\pi\varepsilon_0 r^5}$$

Proposition 5.6.5. *Consider a general distribution of charges contained in some finite volume V. Then the potential can be approximated at large distances by a sum of a monopole potential, dipole pential and some higher order multipole potentials:*

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r} + \frac{\vec{e}\vec{p}}{r^2} + \frac{1}{2r^3}e_i e_j Q_{ij} + \dots\right]$$

*where $\vec{e} = \frac{\vec{x}}{r}$ and $Q = \int_V \rho(\vec{y})\ d^3y$ is the **monopole moment**, $\vec{p} = \int_V \vec{y}\rho(\vec{y})\ d^3y$ is the **dipole moment** and $Q_{ij} = \int_V (3y_iy_j - |\vec{y}|^2\delta_i j)\rho(\vec{y})\ d^3y$ is a higher order **multipole moment**.*

Proof. We start off with the general solution of the potential of an electrostatic system

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}\ d^3y$$

Using the lemma for the Taylor expansion, we have that

$$\begin{aligned}
\phi(\vec{x}) &= \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{y})\left(\frac{1}{|\vec{x}|} + \frac{\vec{x}\cdot\vec{y}}{|\vec{x}|^3} + \frac{3(\vec{x}\cdot\vec{y})^2 - |\vec{x}|^2|\vec{y}|^2}{2|\vec{x}|^5} + \dots\right)\ d^3y \\
&= \frac{1}{4\pi\varepsilon_0}\left[\int_V \frac{\rho(\vec{y})}{r}\ d^3y + \vec{x}\cdot\int_V \frac{\rho(y)\vec{y}}{r^3}\ d^3y + x_ix_j\int_V \frac{(3y_iy_j - \delta_{ij}|\vec{y}|^2)\rho(y)}{2r^5}\ d^3y + \dots\right] \\
&= \frac{1}{4\pi\varepsilon_0}\left[\int_V \frac{\rho(\vec{y})}{r}\ d^3y + \vec{e}\cdot\int_V \frac{\rho(y)\vec{y}}{r^2}\ d^3y + \frac{e_ie_j}{2}\int_V \frac{(3y_iy_j - \delta_{ij}|\vec{y}|^2)\rho(y)}{2r^3}\ d^3y + \dots\right] \\
&= \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r} + \frac{\vec{e}\cdot\vec{p}}{r^2} + \frac{e_ie_j}{2r^3}Q_{ij} + \dots\right]
\end{aligned}$$

□

5.7 Magnetic dipole moments, distant field of a current distribution

Lemma 5.7.1. *Let $F(\vec{y})$ be a function and V a volume in space. Then*

$$\int_V j_i(\vec{y})\partial_i F(\vec{y})\ d^3y = 0$$

Proof. We first note that the continuity equation states that

$$\vec{\nabla}\vec{j} = -\partial_t \rho$$

Since we are working with magnetostatics, ρ is time independent and we have that $\vec{\nabla}\vec{j} = 0$. Hence $\partial_i j_i F(\vec{y})$ for any function $F(\vec{y})$. Now consider the integral over the volume V

$$0 = \int_V \partial_i j_i(\vec{y})F(\vec{y})\ d^3y = -\int_V j_i(\vec{y})\partial_i F(\vec{y})\ d^3y + \oint_{S_V} j_i(\vec{y})F(\vec{y})\ dS_i$$

Since $\rho = 0$ outside the volume V, we must also have that $\vec{j} = \vec{0}$ outside the volume V. This should also be true right on the boundary V_S and hence the surface integral is zero. □

Proposition 5.7.2. *Given a system of currents in a volume V, the vector potential $\vec{A}$ for the magnetic field $\vec{B}$ can be well approximated by the dipole moment*

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{x}}{r^3}$$

Proof. We begin with the general solution for the vector potential

$$\vec{A} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{j}(\vec{y})}{|\vec{x}-\vec{y}|}\ d^3y$$

We can then apply the lemma for the Taylor expansion of $\frac{1}{|\vec{x}-\vec{y}|}$ to obtain

$$\begin{aligned}\vec{A} &= \frac{\mu_0}{4\pi}\int_V \vec{j}\left(\frac{1}{|\vec{x}|} + \frac{y_i x_i}{|\vec{x}|^3} + \dots\right)\ d^3y \\ &= \frac{\mu_0}{4\pi}\left[\frac{1}{|\vec{x}|}\int_V \vec{j}(\vec{y})\ d^3y + \frac{x_i}{|\vec{x}|^3}\int_V y_i\vec{j}(\vec{y})\ d^3y\right]\end{aligned}$$

where we have only considered the first two terms of the expansion. Now consider the function $F(\vec{y}) = y_1$. By the previous lemma, we have that

$$\int_V \partial_i y_1 j_i(\vec{y})\ d^3y = \int_V \delta_{i1} j_i(\vec{y})\ d^3y = \int_V j_1(\vec{y})\ d^3y = 0$$

we can repeat the same process with $F = y_2$ and $F = y_3$ to see tat the integral of any component of the current density is vanishing. Hence the leading term of the Taylor expansion vanishes. We can further simplify the second term by considering the previous lemma with the function $F(\vec{y}) = y_a y_b$. We have that

$$\begin{aligned}
\int_V j_i(\vec{y})\partial_i F(\vec{y})\ d^3y &= \int_V j_i(\vec{y})\partial_i(y_a y_b)\ d^3y \\
&= \int_V j_i(\vec{y})(\delta_{ia}y_b + y_a\delta_{ib})\ d^3y \\
&= \int_V j_a(\vec{y})y_b + j_b(\vec{y})y_a\ d^3y \\
&= 0 \\
&\implies \int_V j_a(\vec{y})y_b\ d^3y = -\int_V j_b(\vec{y})y_a\ d^3y
\end{aligned}$$

Hence in the 3×3 matrix $M_{ab} = \int_V y_b j_a(\vec{y})\ d^3y$ there are only 3 nontrivial elements since the matrix is antisymmetric. They are

$$m_1 = -M_{23}, \quad m_2 = -M_{31}, \quad m_3 = -M_{12}$$

It is therefore easy to see that

$$\begin{aligned}
m_a &= -\frac{1}{2}\epsilon_{abc}M_{bc} \\
&= -\frac{1}{2}\epsilon_{abc}\int_V j_b(\vec{y})y_c\ d^3y \\
\implies \vec{m} &= \frac{1}{2}\int_V \vec{y} \times \vec{j}(\vec{y})\ d^3y
\end{aligned}$$

We therefore have that

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m} \times \vec{x}}{r^3}$$

□

Remark. *The corresponding magnetic field for the magnetic potential derived in the previous proposition is thus*

$$\vec{B} = -\frac{\mu_0}{4\pi}\left(\frac{\vec{m}}{r^3} - 3\frac{\vec{x}(\vec{x}\cdot\vec{m})}{r^5}\right)$$

Example 5.7.3. *Consider a loop of current. Such a current is obviously localised on a contour γ. We can define the current distribution as a line integral*

$$\vec{j} = I\oint_\gamma \delta(\vec{x} - \vec{\gamma}(s))\, d\vec{x}$$

where I is some constant. We can now calculate the magnetic dipole moment of the current loop:

$$\begin{aligned}
m_i &= \frac{1}{2}\int_V y \times \vec{j}(\vec{y})\, d^3y \\
&= \frac{\epsilon_{iab}}{2}\int_V y_a j_b(\vec{y})\, d^3y \\
&= \frac{I\epsilon_{iab}}{2}\int_V y_a \oint_\gamma \delta(\vec{y} - \vec{\gamma}(s))\, dy_b\, d^3y \\
&= \frac{I\epsilon_{iab}}{2}\oint_\gamma \int_V y_a \delta(\vec{y} - \vec{\gamma}(s))\, d^3y\, dy_b \\
&= \frac{I\epsilon_{iab}}{2}\oint_\gamma \int_V y_a \delta(\vec{y} - \vec{\gamma}(s))\, d^3y\, dy_b \\
&= \frac{I\epsilon_{iab}}{2}\oint_\gamma \gamma_a\, dy_b
\end{aligned}$$

For flat contours, this integral gives $|\vec{m}| = IS_\gamma$ where S_γ is the area surrounded by γ. This is the analogue to the electric dipol moment formula for two point-like charges $|\vec{p}| = qd$ where d is the distance between the charges.

5.8 Forces and moments acting on distributions of charges

Consider a system consisting of charges in an external field generated by a continuous charge distribution ρ and current distribution $\vec{j}$. Then the force acting on a small element of this system is the Lorentz force given by

$$d\vec{F}(x) = (d^3x\rho)\vec{E} + (d^3x\vec{j}) \times \vec{B}$$

Definition 5.8.1. *Let $\vec{x}$ be a point in space and $F(\vec{x})$ the force acting at $\vec{x}$. Then we define the **torque** or **rotating moment** of $F(\vec{x})$ to be $\vec{N} = \vec{x} \times \vec{F}(\vec{x})$.*

Continuing with the previous discussion, we have that the torque is

$$d\vec{N}(\vec{x}) = \vec{x} \times d\vec{F}(\vec{x})$$

Proposition 5.8.2. *Consider a system of charges. Then the force acting on the charges due to their own field vanishes.*

Proof. For simplicity's sake, we shall only consider the force generated by the electric field. By the previous discussion, we have that

$$d\vec{F}(\vec{x}) = (d^3x\rho)\vec{E}$$

Dividing both sides by d^3x we have that

$$\frac{d\vec{F}(\vec{x})}{d^3x} = \rho\vec{E}$$

Now integrating both sides with respect to d^3x over a volume V we get

$$\begin{aligned} \vec{F}(\vec{x}) &= \int_V \rho\vec{E}\, d^3x \\ &= \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x})\left(\int_V \frac{\rho(\vec{y})(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3}\, d^3y\right)\, d^3x \\ &= \frac{1}{4\pi\varepsilon_0}\int_V \left(\int_V \frac{\rho(\vec{x})\rho(\vec{y})(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3}\, d^3y\right)\, d^3x \end{aligned}$$

We can see that if we switch the integration variables $\vec{x}$ and $\vec{y}$ we arrive at

$$\frac{1}{4\pi\varepsilon_0}\int_V \left(\int_V \frac{\rho(\vec{x})\rho(\vec{y})(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3}\, d^3y\right)\, d^3x = -\frac{1}{4\pi\varepsilon_0}\int_V \left(\int_V \frac{\rho(\vec{x})\rho(\vec{y})(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3}\, d^3y\right)\, d^3x$$

Hence the integral must be zero and thus the force $\vec{F}(\vec{x})$ must be zero. □

Proposition 5.8.3. *Consider a system of charges with charge density ρ and current density $\vec{j}$ in an external electric field $\vec{E}$ and magnetic field $\vec{B}$. We have that the electrostatic force acting on the system can be approximated by*

$$\vec{F} = Q\vec{E}(0) + \left(\vec{\nabla}\left[\vec{p} \cdot \vec{E}(\vec{x})\right]\right)_{\vec{x}=0}$$

where $\vec{p}$ is the electric dipole moment. Furthermore, the magnetostatic force acting on the system can be approximated by

$$\vec{F} = \left(\vec{\nabla}\left[\vec{m} \cdot \vec{B}(\vec{x})\right]\right)_{\vec{x}=0}$$

where $\vec{m}$ is the magnetic dipole moment.

Proof. We first Taylor expand the electric field around the origin:

$$\vec{E}(\vec{x}) = \vec{E}(0) + x_i \left(\frac{\partial \vec{E}(\vec{x})}{\partial x_i}\right)_{\vec{x}=0} + \dots$$

For a good approximation, we will use only the first two terms of this expansion. Inserting this into the Lorentz force for the electrostatic case, we have that

$$\begin{aligned}
\vec{F} &= \int_V \rho(\vec{x})\vec{E}(\vec{x})\, d^3x \\
&= \vec{E}(0)\int_V \rho(\vec{x})\, d^3x + \left(\frac{\partial \vec{E}(\vec{x})}{\partial x_i}\right)_{\vec{x}=0} \int_V x_i \rho(\vec{x})\, d^3x \\
&= Q_V \vec{E}(0) + p_i \left(\frac{\partial \vec{E}(\vec{x})}{\partial x_i}\right)_{\vec{x}=0} \qquad (5.1)
\end{aligned}$$

Now recall Maxwell's first equation. It says that

$$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$$

But we are working with electrostatics and magnetostatics and thus the fields are time independent, hence $\vec{\nabla} \times \vec{E}$ vanishes. Now consider the vector triple product identity derived in the first section:

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$$

We have that

$$\begin{aligned}
&\vec{p}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{p}\cdot\vec{E}) - \vec{E}(\vec{p}\cdot\vec{\nabla}) \\
&\implies \vec{p}\times(\vec{0}) = \vec{\nabla}(\vec{p}\cdot\vec{E}) - \vec{E}(\vec{p}\cdot\vec{\nabla}) \\
&\implies 0 = \vec{\nabla}(\vec{p}\cdot\vec{E}) - \vec{E}(\vec{p}\cdot\vec{\nabla}) \\
&\implies \vec{\nabla}(\vec{p}\cdot\vec{E}) = \vec{E}(\vec{p}\cdot\vec{\nabla})
\end{aligned} \tag{5.2}$$

The right hand side of Equation (5.2) is exactly the right hand side of Equation (5.1). It therefore follows that the force is

$$\begin{aligned}
\vec{F} &= Q_V\vec{E}(0) + \left(\nabla\left[p_iE_i(\vec{x})\right]\right)_{\vec{x}=0} \\
&= Q_V\vec{E}(0) + \left(\nabla\left[\vec{p}\cdot\vec{E}(\vec{x})\right]\right)_{\vec{x}=0}
\end{aligned}$$

as required.
For the magnetostatic case, we Taylor expand the magnetic field around the origin:

$$\vec{B}(\vec{x}) = \vec{B}(0) + x_i\left(\frac{\partial\vec{B}(\vec{x})}{\partial x_i}\right)_{\vec{x}=0} + \ldots$$

Inserting this into the Lorentz force for the magnetostatic case, we have that

$$\begin{aligned}
F_i &= \left[\int_V \vec{j}(\vec{x})\times\vec{B}(\vec{x})\ d^3x\right]_i \\
&= \left[\int_V \vec{j}(\vec{x})\times\vec{B}(0)\ d^3x + \int_V \vec{j}(\vec{x})\times\left(x_j\left(\frac{\partial\vec{B}(\vec{x})}{\partial x_j}\right)_{\vec{x}=0}\right)\ d^3x\right]_i \\
&= \left[\int_V \vec{j}(\vec{x})\times\vec{B}(0)\ d^3x + \int_V \vec{j}(\vec{x})\times x_j\left(\partial_j\vec{B}(\vec{x})\right)_{\vec{x}=0}\ d^3x\right]_i \\
&= \int_V \epsilon_{iab}j_a(\vec{x})B_b(0)\ d^3x + \int_V \epsilon_{iab}j_a\left(x_j\partial_jB_b(0)\right)\ d^3x \\
&= \int_V \epsilon_{iab}j_a(\vec{x})B_b(0)\ d^3x + \partial_jB_b(0)\int_V \epsilon_{iab}j_ax_j\ d^3x
\end{aligned}$$

Now from previous results, we know the following

$$\int_V j_k(\vec{x})\ d^3x = 0$$
$$\int_V x_a j_k\ d^3x = \epsilon_{akb} m_b$$

where $\vec{m}$ is the magnetic dipole moment. We see that the first term of the expansion of F_i is vanishing and we are left with the following

$$\begin{aligned} F_i &= \epsilon_{iab}\epsilon_{jaz} m_z \partial_j B_b(0) \\ &= \epsilon_{abi}\epsilon_{azj} m_z \partial_j B_b(0) \\ &= m_z \partial_j B_b(0)(\delta_{bz}\delta_{ij} - \delta_{bj}\delta_{iz}) \\ &= m_b \partial_i B_b(0) - m_i \partial_b B_b(0) \end{aligned}$$

But $\vec{\nabla} \cdot \vec{B} = 0$ and hence the second term vanishes. We are thus left with

$$\begin{aligned} F_i &= m_b \partial_i B_b(0) \\ \vec{F} &= \left(\vec{\nabla}\left[\vec{m} \cdot \vec{B}\right]\right)_{\vec{x}=0} \end{aligned}$$

as required. □

Proposition 5.8.4. *Consider a system of charges with charge density ρ and current density $\vec{j}$ in an external electric field $\vec{E}$ and magnetic field $\vec{B}$. We have that the electrostatic torque acting on the system can be approximated by*

$$\vec{N} = \vec{p} \times \vec{E}(0)$$

where $\vec{p}$ is the electric dipole moment. Furthermore, the magnetostatic torque acting on the system can be approximated by

$$\vec{N} = \vec{m} \times \vec{B}(0)$$

where $\vec{m}$ is the magnetic dipole moment.

Proof. We again consider the Taylor expansion of the electric field around the origin:

$$\vec{E}(\vec{x}) = \vec{E}(0) + x_i \left(\frac{\partial \vec{E}(\vec{x})}{\partial x_i}\right)_{\vec{x}=0} + \dots$$

For simplicity's sake, we shall only consider the first term of the expanion. Inserting this into the definition of the torque, we see that

$$\begin{aligned}
N_i &= \left[\int_V \vec{x} \times (\rho(\vec{x})\vec{E}(0))\right]_i \\
&= \int_V \rho(\vec{x})\epsilon_{iab}x_a E_b(0)\ d^3x \\
&= \epsilon_{iab}E_b(0)\int_V \rho(\vec{x})x_a\ d^3x \\
&= \epsilon_{iab}E_b(0)p_a \\
&= [p \times \vec{E}(0)]_i
\end{aligned}$$

as required.
Now we again consider the Taylor expansion of the magnetic field around the origin:

$$\vec{B}(\vec{x}) = \vec{B}(0) + x_i\left(\frac{\partial \vec{B}(\vec{x})}{\partial x_i}\right)_{\vec{x}=0} + \ldots$$

Once more, we shall only consider the first term of the expansion. Inserting it into the definition of torque we obtain

$$\begin{aligned}
N_i &= \left[\int_V \vec{x} \times \left(\vec{j} \times \vec{B}(0)\right)\ d^3x\right]_i \\
&= \int_V \epsilon_{iab}x_a\epsilon_{bxy}j_x B_y(0)\ d^3x \\
&= \epsilon_{iab}\epsilon_{bxy}B_y(0)\int_V x_a j_x\ d^3x \\
&= \epsilon_{iab}\epsilon_{bxy}B_y(0)\epsilon_{axz}m_z \\
&= \epsilon_{iab}B_y(0)\epsilon_{xyb}\epsilon_{xza}m_z \\
&= \epsilon_{iab}B_y(0)(\delta_{yz}\delta_{ba} - \delta_{ya}\delta_{bz})m_z \\
&= \epsilon_{iba}B_z(0)m_z - \epsilon_{iyz}B_y(0)m_z
\end{aligned}$$

Consider $\epsilon_{iba}B_z(0)m_z$. We can rename $a \to b$ and $b \to a$ to get that

$$\begin{aligned}
\epsilon_{iba}B_z(0)m_z &= \epsilon_{iab}B_z(0)m_z \\
&= -\epsilon iab B_z(0)m_z \\
&= 0
\end{aligned}$$

We are thus left with

$$\begin{aligned} N_i &= -\epsilon_{iyz} B_y(0) m_z \\ \vec{N} &= -\vec{B} \times \vec{m} \\ &= \vec{m} \times \vec{B} \end{aligned}$$

as required. □

5.9 Energy of a system in an external field

Consider a system of charges and currents in a volume V under the influence of some fixed external field. As seen in the previous section, there is a force acting on this system. In order to mechanically move the system, we need to apply a force $\vec{F} = -\vec{F}_{ext}$. If we move the system along a very small translation vector $\vec{a} \in \mathbb{R}^3$ then the work done is given by

$$W = \vec{F} \cdot \vec{a} = -\vec{F}_{ext} \cdot \vec{a}$$

Proposition 5.9.1. *Consider a system of charges and currents in a volume V under the influence of some fixed external field. Then the potential energy due to electrostatic forces is given by*

$$U = \int_V \rho(\vec{x}) \phi(\vec{x}) \; d^3x$$

Proof. We start from the definition of work done on the system for a small translation $\vec{x} \to \vec{x} + \vec{a}$:

$$W = -\vec{F}_{ext} \cdot \vec{a}$$

Inserting the electrostatic Lorentz force into the above equation, it follows that

$$\begin{aligned} W &= -\int_V \rho(\vec{x}) \vec{E}(\vec{x}) \; d^3x \cdot \vec{a} \\ &= \int_V \rho(\vec{x}) \vec{\nabla} \phi(\vec{x}) \; d^3x \cdot \vec{a} \end{aligned}$$

Using the integration by parts formula for integrals, we have that

$$\int_V \rho(\vec{x})\vec{\nabla}\phi(\vec{x})\ d^3x = -\int_V \vec{\nabla}\rho(\vec{x})\phi(\vec{x})\ d^3x + \oint_{S_V} \rho(\vec{x})\phi(\vec{x}) \cdot d\vec{S}$$

By assumption, $\rho(\vec{x})$ is vanishing outside of V and therefore must also be vanishing on the boundary of S_V. Thefore the surface integral vanishes and it follows that

$$W = -\int_V \vec{\nabla}\rho(\vec{x})\phi(\vec{x})\ d^3x \cdot \vec{a}$$

We can now use the Taylor expansion $\rho(\vec{x}-\vec{a}) = \rho(\vec{x}) - a_i\partial_i\rho(\vec{x})$, valid for small $|\vec{a}|$ to get

$$\begin{aligned} W &= \int_V (\rho(\vec{x}-\vec{a}) - \rho(\vec{x}))\phi(\vec{x})\ d^3x \\ &= \int_V \rho(\vec{x}-\vec{a})\ d^3x - \int_V \rho(\vec{x})\phi(\vec{x})\ d^3x \end{aligned}$$

Obviously the work done can be expressed as a difference of potential energies $W = U_{after} - U_{before}$ where

$$U = \int_V \rho(\vec{x})\phi(\vec{x})\ d^3x$$

□

Remark. *When the system is small, we can Taylor expand $\phi(\vec{x})$ around the origin to get a good first-order approximation:*

$$\begin{aligned} U &= \int_V \rho(\vec{x})\phi(0) + x_i\partial_i\phi(0))d^3x \\ &= Q\phi(0) + \partial_i\phi(0)\int_V x_i\rho(\vec{x})\ d^3x \\ &= Q\phi(0) + \vec{\nabla}\vec{p}\phi(0) \\ &= \phi(0)\left(Q + \vec{\nabla}\vec{p}\right) \end{aligned}$$

Proposition 5.9.2. *Consider a system of charges and currents in a volume V under the influence of some fixed external field. Then the potential energy due to magnetostatic forces is given by*

$$U = \int_V \vec{j}\cdot\vec{A}\ d^3x$$

Proof. We start with the definition of work done in moving the system by a small translation $\vec{x} \to \vec{x} + \vec{a}$:

$$\begin{aligned}
W &= -\vec{F}_{ext} \cdot \vec{a} \\
&= -\int_V \vec{j}(\vec{x}) \times \vec{B}(\vec{x})\ d^3x \cdot \vec{a} \\
&= -a_i \epsilon_{iab} \int_V j_a(\vec{x}) B_b(\vec{x})\ d^3x \\
&= -a_i \epsilon_{iab} \int_V j_a(\vec{x}) \left[\vec{\nabla} \times \vec{A}(\vec{x})\right]_b\ d^3x \\
&= -a_i \epsilon_{iab} \int_V j_a(\vec{x}) \epsilon_{bxy} \partial_x A_y(\vec{x})\ d^3x \\
&= -a_i \epsilon_{bia} \epsilon_{bxy} \int_V j_a(\vec{x}) \partial_x A_y(\vec{x})\ d^3x \\
&= -a_i (\delta_{ix}\delta_{ay} - \delta_{iy}\delta_{ax}) \int_V j_a(\vec{x}) \partial_x A_y(\vec{x})\ d^3x \\
&= -a_x \int_V j_y \partial_x A_y(\vec{x})\ d^3x + a_y \int_V j_x(\vec{x}) \partial_x A_y(\vec{x})\ d^3x
\end{aligned}$$

We now note that the continuity equation $\vec{\nabla}\vec{j} = -\partial_t \rho$ implies that $\vec{\nabla}\vec{j}$ is vanishing in the time independent magentostatic case. Hence the second term in the above vanishes. We also consider the Taylor expansion for small $|\vec{a}|$, $j_i(\vec{x} - \vec{a}) = j_i(\vec{x}) - a_k \partial_k j_i(\vec{x})$ which allows us to write the first term as

$$\begin{aligned}
W &= -\int_V -a_x \partial_x j_y A_y(\vec{x})\ d^3x \\
&= \int_V (j_y(\vec{x}) - j_y(\vec{x} - \vec{a})) A_y(\vec{x})\ d^3x \\
&= -\int_V (j_y(\vec{x} - \vec{a}) - j_y(\vec{x})) A_y(\vec{x})\ d^3x
\end{aligned}$$

Obviously the work done can be expressed as a difference of potential energies $W = U_{after} - U_{before}$ where

$$U = -\int_V \vec{j} \cdot \vec{A}\, d^3x$$

We now need to take into consideration what happens to the currents after such a translation. We first consider a wire modelled by a contour γ with current I for the current distribution $\vec{j}$. The volume integral for the potential energy hence reduces to a contour integral over the wire:

$$\begin{aligned} U &= -I \oint_\gamma \vec{A} d\vec{l} \\ &= -I \int \vec{\nabla} \times \vec{A} \cdot d\vec{S} \\ &= -I \int \vec{B} \cdot d\vec{S} \\ &= -I\Phi \end{aligned}$$

where S is some surface bounded by γ and Φ is the flux through the surface. When the contour moves through the magnetic field, the Lorentz force acts on the electrons inside the wire which can change the current. This effect is given by **Faraday's Law of Induction**:

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

where $\mathcal{E}$ is the **electromotive force**. In order to fix the current, we need to add some external electromotive force of

$$\mathcal{E}_{ext} = \frac{d\Phi}{dt}$$

We hence need to factor in the work done by this force which is $\mathcal{E}_{ext} dq$ where dq is the charge going through a small section of the wire

$$\begin{aligned} dW &= \frac{d\Phi}{dt} dq \\ &= \frac{dq}{dt} d\Phi \\ &= I d\Phi \\ &= -dU \end{aligned}$$

Hence the total potential energy of the system is $-U = \int_V \vec{j} \cdot \vec{A} \, d^3x$. □

5.10 Self-energy of a system

We now introduce the notion of the energy of a system in its own field.

Definition 5.10.1. *Consider a system of charges and currents in a volume V. We can decompose V into small sections of charges and currents. We define the* ***self-energy*** *of the system to be the sum of the individual energies of the interactions between all these small sections when their size tends to 0.*

Proposition 5.10.2. *Consider a system of charges in a volume V. Then the self-energy of the system in the electrostatic case is*

$$U^{self} = \frac{\varepsilon_0}{2} \int \vec{E} \cdot \vec{E} \, d^3x$$

Proof. Consider a localised system of charges with a fixed charge density $\rho(\vec{x})$ in a volume V. We can divide this system into tiny elements which we label either i or k. Each of these small elements can be described by its own density $\rho_i(\vec{x})$ and hence the total density is the sum

$$\rho(\vec{x}) = \sum_i \rho_i(\vec{x})$$

Now denote the potential energy of the element i created in the external field generated by an element k by U_{ik}. Let V_i represent the small volume that i occupies. From the previous section, we have that

$$\begin{aligned} U_{ik} &= \int_{V_i} \rho_i(\vec{x})\phi_k(\vec{x}) \\ &= \int_{V_i} \int_{V_k} \frac{\rho_i(\vec{x})\rho_k(\vec{y})}{4\pi\varepsilon_0|\vec{x}-\vec{y}|} \, d^3y \, d^3x \end{aligned}$$

We can see that $U_{ik} = U_{ki}$. Now from the definition of self-energy, we have that

$$U^{self} = \lim_{size \to 0} \sum_{i<k} U_{ik} \tag{5.3}$$

We require $i < k$ to avoid double counting elements. We note that

$$\sum_{i,k} U_ik = 2 \sum_{i<k} U_{ik} + \sum_i U_{ii}$$

Therefore we can rewrite Equation (5.3) as follows:

$$U^{self} = \frac{1}{2} \lim_{size \to 0} \left(\sum_{i,k} U_{ik} - \sum_i U_{ii} \right)$$

The first sum can be written as

$$\begin{aligned} \sum_{i,k} U_{ik} &= \sum_{i,k} \int_{V_i} \int_{V_k} \frac{\rho_i(\vec{x})\rho_k(\vec{y})}{4\pi\varepsilon_0 |\vec{x} - \vec{y}|} \, d^3y \, d^3x \\ &= \int_{V_i} \int_{V_k} \frac{\sum_i (\rho_i(\vec{x})) \sum_k (\rho_k(\vec{y}))}{4\pi\varepsilon_0 |\vec{x} - \vec{y}|} \, d^3y \, d^3x \\ &= \int_V \int_V \frac{\rho(\vec{x})\rho(\vec{y})}{4\pi\varepsilon_0 |\vec{x} - \vec{y}|} \, d^3y \, d^3x \end{aligned}$$

This quantity does not depend on the way the system is divided and therefore the limit can be dropped. Now the second term is

$$\sum_i U_{ii} = \sum_i \int_{V_i} \int_{V_i} \frac{\rho_i(\vec{x})\rho_i(\vec{y})}{4\pi\varepsilon_0 |\vec{x} - \vec{y}|} \, d^3y \, d^3x$$

We claim that this integral is actually vanishing in the limit $size \to 0$. To see this, assume that V_i are tiny cubes of sides a. We can and will assume that the charge density $\rho(\vec{x}) < C$ for some constant C. Now since each V_i are small, we can assume that there exists an $\vec{x}_i$ such that $\rho_i(\vec{x}) = \rho(\vec{x}_i)$. It follows that the integral, in the limit, is:

$$\begin{aligned} \int_{V_i} \int_{V_i} \frac{\rho_i(\vec{x})\rho_i(\vec{y})}{4\pi\varepsilon_0 |\vec{x} - \vec{y}|} \, d^3y \, d^3x &\simeq \frac{\rho(\vec{x}_i)^2}{4\pi\varepsilon_0} \int_{V_i} \int_{V_i} \frac{1}{|\vec{x} - \vec{y}|} \, d^3y \, d^3x \\ &\simeq \frac{\rho(\vec{x}_i)^2}{4\pi\varepsilon_0} a^5 \end{aligned}$$

where the a^5 term was introduced by estimating the integral. Each integral $\int_{V_i} d^3x$ contributes a^3 as it is the volume of the cube V_i. The $\frac{1}{|\vec{x}-\vec{y}|}$ has maximal value $\frac{1}{a}$. Indeed if $\vec{x} - \vec{y} = (a, 0, 0)$. Then $\frac{1}{|\vec{x}-\vec{y}} = \frac{1}{a}|$. We hence obtain a term a^5. We now have to sum up each integral which is the same as multiplying by the number of cubes. The number of cubes is given by $\frac{V}{a^3}$.

Hence

$$\begin{aligned}\lim_{a\to 0}\sum_i U_{ii} &= \lim_{a\to 0}\frac{\rho(\vec{x}_i)^2 V a^2}{4\pi\varepsilon_0}\\ &< \lim_{a\to 0}\frac{C^2 V a^2}{4\pi\varepsilon_0}\\ &= 0\end{aligned}$$

Thus this term vanishes and we are left with

$$\begin{aligned}U^{self} &= \frac{1}{2}\int_V\int_V \frac{\rho(\vec{x})\rho(\vec{y})}{|\vec{x}-\vec{y}|}\ d^3y d^3x\\ &= \frac{1}{2}\int_V \rho(\vec{x})\phi(\vec{x})\ d^3y\, d^3x\end{aligned}$$

Now using the formulation for scalar potential $-\Delta\phi(\vec{x}) = \frac{\rho}{\varepsilon_0}$ we have

$$\begin{aligned}U^{self} &= -\frac{\varepsilon_0}{2}\int_V \phi(\vec{x})\partial_i\partial_i\phi(\vec{x})\ d^3x\\ &= \frac{\varepsilon_0}{2}\int_V \partial_i\phi(\vec{x})\partial_i\phi(\vec{x})\ d^3x\\ &= \frac{\varepsilon_0}{2}\int_V \vec{E}\cdot\vec{E}\ d^3x\end{aligned}$$

where we integrated by parts and dropped the vanishing surface integral as usual. □

Proposition 5.10.3. *Consider a system of currents in a volume V. Then the self-energy of the system in the magnetostatic case is*

$$U^{self} = \frac{1}{2\mu_0}\int \vec{B}\cdot\vec{B}\ d^3x$$

Proof. The proof is left as an exercise to the reader. It follows the same argumentation as the previous proof. □

Remark. *If we have a system of both charges and currents, we have the following general formula for the self-energy:*

$$U^{self} = \frac{\varepsilon_0}{2}\int_V \left[\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B}\right]$$

Chapter 6

Time-dependent fields

If $\vec{E}$ or $\vec{B}$ are time dependent, we can no longer decouple electric and magnetic phenomena like we have been doing previously. This is evident from the two Maxwell equations

$$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$$
$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2}\partial_t \vec{E} + \mu_0 \vec{j}$$

Obviously, $\vec{E}$ and $\vec{B}$ depend on each other. Another consequence of this is that electromagnetic waves can easily propgate through space even when $\rho(\vec{x}) = 0$ and $\vec{j}(\vec{x}) = 0$ (such as in vacuum).

6.1 Electromagnetic waves

Definition 6.1.1. *We define $\Box$ to be the linear* ***wave operator****:*

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$$

Given a function $f(\vec{x}, t)$, the differential equation

$$\Box f = 0$$

is called the ***wave equation****.*

Remark. *Given a scalar potential ϕ and vector potential $\vec{A}$ satisfying the Lorentz condition we can reformulate Maxwell's equations as follows:*

$$\Box\vec{E} = \frac{\rho(\vec{x})}{\varepsilon_0} \tag{6.1}$$

$$\Box\vec{B} = \mu_0\vec{j}(\vec{x}) \tag{6.2}$$

Proposition 6.1.2. *Consider a system with $\rho(\vec{x}) = 0$ and $\vec{j}(\vec{x})$. Then $\vec{E}$ and $\vec{B}$ satisfy the wave equation.*

Proof. This follows from the previous remark. □

6.2 One-dimension wave equation

The one-dimensional wave equation refers to one spatial dimension. For a function $\psi(x,t)$, the one-dimensional wave equation is

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\psi(\vec{x},t) = 0 \tag{6.3}$$

where c is a positive velocity. This equation has two special solutions, namely

$$\psi_{\pm}^{(k)}(x,t) := sin(kx \pm \omega t)$$

By plugging $\psi_{\pm}$ into Equation (6.3), we see that they are solutions if and only if $\omega = ck$. $\psi_{-}^{(k)}$ represents a sine wave moving in the positive x direction with velocity c. The solution $\psi_{+}^{(k)}$ represents a sine wave moving in the negative x direction with velocity $-c$. We define the **wave length** λ to be the distance of two wave crests in the graph $\psi_{\pm}^{(k)}(x,t_0)$ at a fixed time t_0: $\lambda = \frac{2\pi}{k}$. The quantity k is referred to as the **wave number** and ω is the **frequency**.
Since the wave operator is a linear operator, the principle of superposition of applies to the wave equation and linear combinations of solutions are again solutions. Consider the following

$$\tilde{\psi}_{\pm}^{(k)}(x,t) := cos(kx \pm \omega t)$$

with the same relation between k and ω as before. It follows that the complex superpositions

$$\begin{aligned}\Psi_{\pm}^{(k)}(x,t) &:= \tilde{\psi}_{\pm}^{(k)}(x,t) + i\psi_{\pm}^{(k)}(x,t)\\ &= e^{i(kx\pm\omega t)}\end{aligned}$$

are again solutions. Such solutions are called **monochromatic waves** as they only consist of one frequency ω. Any square integrable solution $\phi(x,t)$ to the wave equation can be written as the superposition of monochromatic waves:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\hat{\psi}_+(k)e^{i(kx+\omega(k)t)}+\hat{\psi}_-^{i(kx-\omega(k)9t)}\right)\,dk$$

Each $\omega(k)$ is determined by k and speed c. The functions $\hat{\psi}_\pm(k)$ are the Fourier transforms of $\psi_\pm(x,t)$ and they determine how strongly $\Psi_\pm^{(k)}$ contribue to $\psi_\pm(x,t)$. The solution $\psi(x,t)$ is unique once we pose initial conditions on the system:

$$\psi(x,t=0)=h(x)$$
$$\frac{\partial}{\partial t}\psi(x,t=0)=g(x)$$

6.3 Three-dimensional wave equation

We can extend the solutions found in the previous section for the three-dimensional wave operator $\Box=\frac{1}{c^2}\frac{\partial^2}{\partial^2 t}-\Delta$.
Let $f:\mathbb{R}\to\mathbb{C}$ be a twice differentiable function and $\vec{k}\in\mathbb{R}^3$. Denote $k:=|\vec{k}|$. Then

$$\psi_f^{(\vec{k})}(\vec{x},t)=f(\vec{k}\cdot\vec{x}-\omega_{\vec{k}}t)$$

solves $\Box\psi=0$ if and only if $\omega_{\vec{k}}=ck$. Such solutions are called **plane waves** with **wave vector** $\vec{k}$ and frequency ω.
Another important type of solution is given by **spherical waves**

$$\psi_f^{(k)}(\vec{x},t):=\frac{f(kx-\omega t)}{r} \tag{6.4}$$

where $k\in\mathbb{R}$ and $r:=|\vec{x}|$ is the radial distance from the origin. For fixed time, the value of $\psi_f^{(k)}$ only depends on the distance of $\vec{x}$ from the origin. As time elapses, the profile given by f spreas radially outward with speed c, thereby decreasing in amplitude because of the $\frac{1}{r}$ factor.
If we replace the minus sign in Equation (6.4) with a plus sign, we obtain incoming spherical waves that converge towards the origin.

Similar to the one dimensional case, plane waves with $f(x) = e^x$ are monochromatic plane waves. Again, any square integrable solution can be written as the superposition of monochromatic plane waves through Fourier decomposition:

$$\frac{1}{(2\pi)^3}\iiint \ldots d^3x$$

with $\omega_{\vec{k}} = ck$.

In the case of electromagnetic waves propogating through a vacuum, we have six three dimensional waves equations, one for each component of the electric and magnetic fields

$$\Box\vec{E}(\vec{x},t) = 0$$
$$\Box\vec{B}(\vec{x},t) = 0$$

In addition, $\vec{E}$ and $\vec{B}$ have to satisfy Maxwell's equations. Since $\rho = 0$ and $\vec{j} = 0$, we have that

$$\vec{\nabla}\cdot\vec{E} = 0$$
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad (6.5)$$

We shall focus on monochromatic plane waves and we can write

$$\vec{E}(\vec{x},t) = Re(\vec{E}_0 e^{i(\vec{k}\cdot\vec{x}+\omega_{\vec{k}}t)})$$
$$\vec{B}(\vec{x},t) = Re(\vec{B}_0 e^{i(\vec{k}\cdot\vec{x}+\omega_{\vec{k}}t)}) \qquad (6.6)$$

where $\vec{E}_0$ and $\vec{B}_0$ are constant vectors. They determine the **polarisation** of the waves - the direction of oscillation. Inserting Equations (6.6) into Equations (6.5) we see that

$$\vec{E}_0\cdot\vec{k} = 0$$
$$\vec{B}_0\cdot\vec{k} = 0$$

This implies that in plane electromagnetic waves, the electric and magnetic field vectors are perpindicular to the direction of propogation (they are **transverse** waves as opposed to **longitudinal** waves such as sound waves).

Now the other two Maxwell's Equations for waves in a vacuum say that

$$\vec{\nabla}\times\vec{E} = -\partial_t\vec{B}$$
$$\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\partial_t\vec{E} \qquad (6.7)$$

Inserting Equations (6.6) into Equations (6.7) we see that

$$\vec{k} \times \vec{E}_0 = \omega \vec{B}_0$$

Because of the relation $\omega = ck$, we have that

$$\vec{B}_0 = \frac{1}{c} \frac{\vec{k}}{k} \times \vec{E}_0$$

This shows that the electric and magnetic fields are both perpendicular to $\vec{k}$ and to each other.

6.4 Energy and momentum in electrodynamics

Consider a system of volume V with charge density $\rho(\vec{x}, t)$ and velocity field $\vec{v}(\vec{x}, t)$ which gives the current distribution $\vec{j}(\vec{x}, t)$. We want to compute the rate of change with respect to time of the energy E_V contained in V (also called the **power**). This receives contributions from the mechanical power of the moving charges contained in V and also from the field power.
We begin with the former. Consider the mechanical power of a point particle:

$$\frac{d}{dt} E = \vec{v} \cdot \vec{F}$$

where $\vec{F}$ is the Lorentz force acting on the particle. We have that

$$\begin{aligned}
\frac{dE_V^{mech}}{dt} &= \int_V \vec{v}(\vec{x}, t) \cdot \left[\rho(\vec{x}, t) \vec{E}(\vec{x}, t) + \vec{j}(\vec{x}, t) \times \vec{B}(\vec{x}, t) \right] \, d^3x \\
&= \int_V \rho(\vec{x}, t) \left[\vec{v}(\vec{x}, t) \cdot \vec{E}(\vec{x}, t) \right] \, d^3x \\
&= \int_V \vec{j}(\vec{x}, t) \cdot \vec{E}(\vec{x}, t) \, d^3x
\end{aligned}$$

where we have used the fact that $\vec{j}(\vec{x},t) \times \vec{B}(\vec{x},t)$ is perpendicular to $\vec{v}(\vec{x},t)$. Using the formula for self-energy, we have that

$$\begin{aligned}
\frac{dE_V^{field}}{dt} &= \frac{d}{dt}\int_V \left[\frac{\varepsilon_0}{2}\vec{E}\cdot\vec{E} + \frac{1}{2\mu_0}\vec{B}\cdot\vec{B}\right]\, d^3x \\
&= \int_V \left[\varepsilon_0 \vec{E}\cdot\left(\frac{\partial}{\partial t}\vec{E}\right) + \frac{1}{\mu_0}\vec{B}\cdot\left(\frac{\partial}{\partial t}\vec{B}\right)\right]\, d^3x \\
&= \int_V \left[\varepsilon_0 c^2 \vec{E}\cdot\left(\vec{\nabla}\times\vec{B} - \mu_0\vec{j}\right) - \frac{1}{\mu_0}\vec{B}\cdot\left(\vec{\nabla}\times\vec{E}\right)\right]\, d^3x \\
&= -\int_V \vec{E}\cdot\vec{j}\, d^3x + \frac{1}{\mu_0}\int_V \left[\vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right) - \vec{B}\cdot\left(\vec{\nabla}\times\vec{E}\right)\right]\, d^3x \\
&= -\int_V \vec{E}\cdot\vec{j}\, d^3x - \frac{1}{\mu_0}\int_V \vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right)\, d^3x \\
&= -\int_V \vec{E}\cdot\vec{j}\, d^3x - \frac{1}{\mu_0}\oint_{S_V} \vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right)\cdot d\vec{S}
\end{aligned}$$

where we have used Maxwell's Equations, the triple scalar product identity and the divergence theorem.

Definition 6.4.1. *Given an electric field $\vec{E}$ and a magnetic field $\vec{B}$, we define the **Poynting vector** $\vec{S}_{Poy}$ to be*

$$\vec{S}_{Poy} := \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

It is thus clear from the previous deductions that

$$\frac{d}{dt}\left(E_V^{mech} + E_V^{field}\right) = -\oint_{S_V} \vec{S}_{Poy}\cdot d\vec{S} \tag{6.8}$$

We can see that if we make V bigger until $V \to \mathbb{R}^3$ and assume that the electric and magnetic fields decay at large distances, the circulation of the Poynting vector vanishes and we get that the total energy of the system is conserved in time.

If we apply the divergence theorem to Equation (6.8), we arrive at the differential equation

$$\frac{\partial}{\partial_t}w + \vec{\nabla}\cdot\vec{S}_{Poy} = -\vec{j}\cdot\vec{E}$$

where $w = \frac{\varepsilon_0}{2}\vec{E}\cdot\vec{E} + \frac{1}{2\mu_0}\vec{B}\cdot\vec{B}$. In the absence of moving charges, we obtain a conservation law.
Conservation of momentum in electrodynamics can be shown in a very similar way. We start with

$$\frac{d}{dt}\vec{p} = \int_V \vec{f}\, d^3x$$

where $\vec{f}$ is the Lorentz force.

Definition 6.4.2. *Given an electric field $\vec{E}$ and magnetic field $\vec{B}$, we define the **momentum density** $\vec{g}$ to be*

$$\vec{g} := \frac{1}{c^2}\vec{S}_{Poy}$$

6.5 Fields genererated by time-dependent charge and currents

Consider time dependent densities $\rho(\vec{x},t)$ and $\vec{j}(\vec{x},t)$. We are interested in the electric and magnetic fields they generate. We can once again decouple the differential equations using the scalar and vector potentials:

$$\begin{aligned} \Box\phi &= \frac{\rho}{\varepsilon_0} \\ &= \Box\vec{A} = \mu_0\vec{j} \end{aligned}$$

As before, we can construct a Green's function $G_\Box$ for the wave operator $\Box$ which satisfies

$$\Box G_\Box = \delta^{(1)}(t-t')\delta^{(3)}(\vec{x}-\vec{x}')$$

where on the right hand side, we have a one-dimension delta distribution for the time argument multiplied by a three dimensional delta distribution for the spatial arguments. It turns out that

$$G_\Box = \frac{1}{4\pi}\frac{\delta^{(1)}(t'-t+\frac{|\vec{x}'-\vec{x}|}{c})}{|\vec{x}'-\vec{x}|}$$

This is referred to as the **retarded Green's function**. The description of retarded follows from considering the potentials:

$$= \phi(\vec{x}, t) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\vec{x}, t_R)}{|\vec{x}' - \vec{x}|}\, d^3x'$$

$$= \vec{A}(\vec{x}, t) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{j}(\vec{x}, t_R)}{|\vec{x}' - \vec{x}|}\, d^3x$$

where $t_R := t - \frac{|\vec{x}' - \vec{x}|}{c}$ and is referred to as **retarded time**. The potentials at $(\vec{x}, t)$ depend all space points $\vec{x}'$ but at an earlier time t_R. The difference between t and t_R is precisely the time an electromagnetic wave takes to travel from the point $\vec{x}'$ to $\vec{x}$.

Chapter 7

Special Relativity

7.1 Main ideas and postulates

Definition 7.1.1. *We define an* ***inertial frame of reference*** *to be a system of observers able to report and record the time of certain events which happen close to them. We assume that the observers within the given frame of reference do not move relatively to each other and there is an observer relatively close to any point in space for each frame of reference. We also assume that all observers within a given frame of reference have their clocks synchronised.*

Assume there are two frames of reference moving with the relative speed u along the x-axis. We can synchronise their clocks by setting them both to the same time when the two frames of reference coincide. Obviously if there is an event at some (x, y, z, t) in one frame then in the other frame's coordinate system, the event is at $(x' = x - ut, y' = y, z' = z, t' = t)$.

Definition 7.1.2. *Consider two frames of refence with coordinate systems* (x, y, z, t) *and* (x', y', z', t')*. Let an event occur in one frame of reference at the point* (x, y, z, t)*. Then the transformation*

$$(x, y, z, t) \to (x - ut, y, z, t)$$

is called the ***Galilean transformation*** *and gives the coordinates of the event in the other frame of reference.*

The principle of relativity essentially states that since space-time is homogeneous, we expect all physical laws to be the same no matter at what

point in space the law is acting. Newtonian mechanics is obviously compatible with this principle. Indeed, given any particle which is moving according to the laws of Newton in frame at some constant acceleration $\vec{a}$, we have that $m\vec{a} = \vec{F}$ where $\vec{F}$ is the force acting on the particle. The acceleration $\vec{a}$ is obviously invariant under a space-time translation and hence Newton's laws are the same in all inertial frames.
On the other hand, Maxwell's equations do not seem to agree with the principle of relativity. Consider the one-dimensional wave operator acting on the scalar potential:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi = 0 \tag{7.1}$$

By the chain rule, we have that the Galilean transformation satisfies

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} \\ &= \frac{\partial}{\partial x'} \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial t'} &= \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} \\ &= \frac{\partial x}{\partial t}\frac{\partial}{\partial x} + \frac{\partial t}{\partial t}\frac{\partial}{\partial t} \\ &= u\frac{\partial}{\partial x} + \frac{\partial}{\partial t} \end{aligned}$$

It therefore follows that

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial x'^2}$$

$$\frac{\partial^2}{\partial t^2} = \left(\frac{\partial}{\partial t'} - u\frac{\partial}{\partial x'}\right)^2$$

Thus Equation (7.1) transforms as follows

$$\left(\frac{1}{c^2}\left(\frac{\partial}{\partial t'} - u\frac{\partial}{\partial x'}\right)^2 - \frac{\partial^2}{\partial x'^2}\right)\phi = 0$$

We see that the wave equation is variant under Galilean transformation and hence Maxwell's equations are not preserved. In particular, inside a moving space ship, all electromagnetic phenomena should be different from those in a stationary ship. One of the consequences of Maxwell's equations is that electromagnetic waves propogate in all directions equally and at the same speed c. According to the Galilean transformations, the speed of light as measured inside the moving space ship should be $c - u$. However many experiments have been done and they all find that the speed of light c is constant inside such a moving intertial frame. The only way to solve this contradiction is to assume that Newton's laws are wrong.

7.2 Time Dilation

We show that in order for the principle of relativity to hold for electromagnetic phenomena, all processes taking place in a moving coordinate system will be detected as occuring slower in a static coordinate system.
We can consider a simple system of two perfect mirrors A and B seperated by a distance d between which a beam of light bounces. We can define a clock on this system as follows: everytime the light beam hits the mirror B, the clock ticks. Since light must travel at the same speed c in all reference frames, this gives us the following period for the clock:

$$\Delta t = \frac{2d}{c}$$

Now consider another identical system moving at a speed v relative to the original resting one. Obviously, the period of the light in the moving clock is

$$\Delta t' = \frac{2D}{c} \tag{7.2}$$

where D is the length of the side of the triangle that the path of the light forms. By Pythagoras theorem, it is easy to see that

$$D = \sqrt{\left(\frac{1}{2}u\Delta t'\right)^2 + d^2}$$

Substituting this into Equation (7.2) we have that

$$\Delta t' = \frac{2\sqrt{\left(\frac{1}{2}u\Delta t'\right)^2 + d^2}}{c}$$

Solving for $\Delta t'$ it follows that

$$\Delta t' = \frac{\frac{2d}{c}}{\sqrt{1 - \left(\frac{u^2}{c^2}\right)}}$$

Now subtituting in the definition of Δt, we have the following equation

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \left(\frac{u^2}{c^2}\right)}}$$

7.3 Length Contraction

Consider the same clock as in the previous section except rotated so the light beam travels parallel to the direction of motion. We have, once again

$$\Delta t = \frac{2d_0}{c}$$

for the period of the clock where d_0 is the seperaton of the mirrors in the static frame of reference. We show that in the moving frame, the length of the seperation contracts. Let d denote the length of the seperation in the moving reference frame. To an observer that sees the clock pass at a velocity u, the light takes more time to traverse the seperation when the wave is travelling in the same direction as the frame. It takes less time to traverse the seperation when travelling in the opposite direction. We have that

$$\Delta t' = \frac{d}{c+u} + \frac{d}{c-u}$$

Now from the previous section, we know that

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \left(\frac{u^2}{c^2}\right)}}$$

Using the three previous equations, we can eliminate Δt and Δt_0 and we are left with

$$d = d_0\sqrt{1 - \left(\frac{u^2}{c^2}\right)}$$

Hence $d < d_0$ and lengths parallel to the direction of travel contract.

7.4 Formal Derivation of the Lorentz Transformation

Convention 7.4.1. *Consider a coordinate system* (ct, x, y, z) *we shall denote each coordinate by*

$$x^0 \equiv ct, x^1 \equiv x, x^2 \equiv y, x^3 \equiv z$$

From now on, we shall use superscripts to denote indicies and not powers (unless explicitly stated otherwise).

Consider two references frames S and S' with coordinate systems (x^0, x^1, x^2, x^3) and (x'^0, x'^1, x'^2, x'^3). We want to find four functions $f^\mu, \mu = 0, 1, 2, 3$ such that

$$x'^\mu = f^\mu(x^0, x^1, x^2, x^3)$$

Such transformations should be invertible.

Both frames are inertial which means that free bodies move along straight lines with a constant speed. Hence

$$x^i = A^i x^0 + B^i$$
$$\downarrow$$
$$x'^i = A'^i x'^0 + B'^i$$

for some constants A^i, B^i, A'^i, B'^i. The only non-singular functions that preserves straight lines are linear functions of the form

$$f^\alpha(x^\beta) = \Lambda^\alpha_\beta + b^\alpha$$

where Λ^α_β is some linear map and b^α is some constant. The repeated Greek indices are summed over $0, 1, 2, 3$. These depend only on the relative velocity u between the two frames

We can choose S and S' such that their origins coincide at $t = t' = 0$. This means that

$$f^\alpha(0, 0, 0, 0) = 0$$

for $\alpha = 0, 1, 2, 3$. We see that in the above function we derived, $b^\alpha = 0$.

We define S' such that the $y' =$ plane displaces along itself and coincides with the $y = 0$ plane. This means that for any x, z and t, we should have

$$y' = \Lambda^2_0 ct + \Lambda^2_1 x + \Lambda^2_2 0 + \Lambda^3_3 z = 0$$

for all x, z, t. This implies that $\Lambda^2_0 = \Lambda^2_1 = \Lambda^2_3 = 0$. We can apply a similar argument to the $z' = 0$ plane to arrive at $\Lambda^3_0 = \Lambda^3_1 = \Lambda^3_2 = 0$. At the moment, we have the following matrix:

$$\Lambda^\mu_\nu = \begin{pmatrix} \Lambda^0_0 & \Lambda^0_1 & \Lambda^0_2 & \Lambda^0_3 \\ \Lambda^1_0 & \Lambda^1_1 & \Lambda^1_2 & \Lambda^1_3 \\ 0 & 0 & \Lambda^2_2 & 0 \\ 0 & 0 & 0 & \Lambda^3_3 \end{pmatrix}$$

The plane $x' = 0$, as seen from S, moves to the right with a constant speed u. Therefore the plane $x' = 0$ should be mapped to the plane $x = ut$:

$$x' = \Lambda^1_0 ct + \Lambda^1_1 ut + \Lambda^1_2 y + \Lambda^1_3 z = 0$$

for all y, z, t. This implies that $\Lambda^1_2 = \Lambda^1_3 = 0$. To summarise, we now have:

$$\Lambda^\mu_\nu = \begin{pmatrix} \Lambda^0_0 & \Lambda^0_1 & \Lambda^0_2 & \Lambda^0_3 \\ -\frac{u}{c}\Lambda^1_0 & \Lambda^1_1 & 0 & 0 \\ 0 & 0 & \Lambda^2_2 & 0 \\ 0 & 0 & 0 & \Lambda^3_3 \end{pmatrix}$$

Switching perspectives, the $x = 0$ plane, seen from S, moves to the left with the same speed u. Hence $x = 0$ should be mapped to $x' = -ut$:

$$\begin{aligned} ut' &= x' = \Lambda^1_0 ct \\ ct' &= \Lambda^0_0 + \Lambda^0_1 0 + \Lambda^0_2 y + \Lambda^0_3 z \end{aligned}$$

Combining these two, it follows that

$$\Lambda^1_0 \frac{ct}{u} = \Lambda^0_0 + \Lambda^0_2 y + \Lambda^0_3 z$$

for all t, y, z. This is only possible if $\Lambda^0_2 y = \Lambda^0_3 z = 0$ and $\Lambda^1_0 = -\frac{u}{c}\Lambda^0_0$ which implies that $\Lambda^1_1 = \Lambda^0_0$. Hence we now have:

$$\Lambda^\mu_\nu = \begin{pmatrix} \Lambda^0_0 & \Lambda^0_1 & 0 & 0 \\ -\frac{u}{c}\Lambda^0_0 & \Lambda^0_0 & 0 & 0 \\ 0 & 0 & \Lambda^2_2 & 0 \\ 0 & 0 & 0 & \Lambda^3_3 \end{pmatrix}$$

We should also impose the constancy of the speed of light. Any reay of light travelling in a direction $\vec{n}$ should be mapped to another ray of light travelling in a possibly different direction. We require that $\vec{x} = \vec{n}ct$ becomes $\vec{x}' = \vec{n}'ct'$. Obviously,

$$\begin{aligned} n'^i = \frac{x'^i}{x'^0} &= \frac{\Lambda^i_0 x^0 + \Lambda^i_j x^j}{\Lambda^0_0 x^0 + \Lambda^0_j x^j} \\ &= \frac{\Lambda^i_0 x^0 + \Lambda^i_j x^j}{\Lambda^0_0 x^0 + \Lambda^0_j x^j} \\ &= \frac{\Lambda^i_0 x^0 + \Lambda^i_j n^j x^0}{\Lambda^0_0 x^0 + \Lambda^0_j n^j x^0} \\ &= \frac{\Lambda^i_0 + \Lambda^i_j n^j}{\Lambda^0_0 + \Lambda^0_j n^j} \end{aligned}$$

First consider $\vec{n} = (1, 0, 0)$, then

$$\begin{aligned} n'^i &= \frac{\Lambda^i_0 + \Lambda^i_j n^j}{\Lambda^0_0 + \Lambda^0_j n^j} \\ &= \frac{\Lambda^i_0 + \Lambda^i_1}{\Lambda^0_0 + \Lambda^0_1} \end{aligned}$$

We see that $n'^2 = n'^3 = 0$. Hence $n'^1 = 1$ which gives us $\Lambda^0_1 = \Lambda^1_0$. Now consider $\vec{n} = (0, 1, 0)$:

$$\begin{aligned} n'^i &= \frac{\Lambda^i_0 + \Lambda^i_j n^j}{\Lambda^0_0 + \Lambda^0_j n^j} \\ &= \frac{\Lambda^i_0 + \Lambda^i_2}{\Lambda^0_0 + \Lambda^0_2} \\ &= \frac{\Lambda^i_0 + \Lambda^i_2}{\Lambda^0_0} \end{aligned}$$

It follows that $n'^1 = -\frac{u}{c}, n'^2 = \frac{\Lambda^2_2}{\Lambda^0_0}, n'^3 = 0$. Since $\vec{n}'$ is a unit vector, we have that

$$1 = \sqrt{\left(-\frac{u}{c}\right) + \left(\frac{\Lambda^2_2}{\Lambda^0_0}\right)}$$

whence it follows that

$$\Lambda^2_2 = \Lambda^0_0 \sqrt{1 - \left(\frac{u^2}{c^2}\right)}$$

The same argumentation for $\vec{n} = (0, 0, 1)$ gives us

$$\Lambda^3_3 = \Lambda^0_0 \sqrt{1 - \left(\frac{u^2}{c^2}\right)}$$

and therefore

$$\Lambda^\mu_\nu = \begin{pmatrix} \Lambda^0_0 & -\frac{u}{c}\Lambda^0_0 & 0 & 0 \\ -\frac{u}{c}\Lambda^0_0 & \Lambda^0_0 & 0 & 0 \\ 0 & 0 & \Lambda^0_0\sqrt{1 - \left(\frac{u^2}{c^2}\right)} & 0 \\ 0 & 0 & 0 & \Lambda^0_0\sqrt{1 - \left(\frac{u^2}{c^2}\right)} \end{pmatrix}$$

Since we assumed that $y = 0$ maps to $y' = 0$ and $z = 0$ maps to $z' = 0$, we have that

$$1 = \Lambda^0_0 \sqrt{1 - \left(\frac{u^2}{c^2}\right)}$$
$$\Lambda^0_0 = \frac{1}{\sqrt{1 - \left(\frac{u^2}{c^2}\right)}}$$

Hence we have the final result

$$\Lambda^\mu_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\gamma = \frac{1}{\sqrt{1-\left(\frac{u^2}{c^2}\right)}}$ and $\beta = \frac{u}{c}$.

7.5 Non-Relativistic Limit

In the limit $v << c$, we expect to recover the usual Newtonian physics. In order to show this, we Taylor expand γ:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \mathcal{O}\left(\frac{v^4}{c^4}\right)$$

Since the speed of light is very large, the above expansion is extremely accurate. We can thus write the Lorentz transformations as

$$ct' = \gamma(ct - \beta x) \simeq \left(1 - \frac{v^2}{c^2}\right)\left(ct - \frac{v}{c}x\right) \simeq ct - \frac{v^3}{x^3}x \simeq ct$$
$$x' = \gamma(x - \beta ct) \simeq \left(1 - \frac{v^2}{c^2}\right)\left(x - \frac{v}{c}ct\right) \simeq x - vt$$

We therefore recover the usual Galilean transformations.

7.6 Lorentz Transformation as a Rotation by a a Hyperbolic Angle

Consider two reference frames moving at a relative velocity u. We define the **rapidity** variable θ such that

$$\tanh\theta = \frac{u}{c}$$

The hyperbolic identity

$$\cosh\theta = \frac{1}{\sqrt{1-\tanh^2\theta}}$$

implies that

$$\gamma = \cosh\theta, \beta\gamma = \sinh\theta$$

Hence we have that the Lorentz transformation satisfies

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

We can also convert the hyperbolic functions to trigonmetric functions to see that the Lorentz transformation is a rotation by an imaginary angle $i\theta$.

7.7 Relativity of Simultaneity

Consider two reference frames S and S' and let P_1 and P_2 be two events in S with coordinates

$$P_1 = (t, x^1, x^2, x^3), P_2 = (t, x'^1, x'^2, x'^3)$$

Due to the Lorentz transformation, these events do not occur simultaneously in S':

$$P_1 = (t\gamma - \beta\gamma x^1, x^2, x^3), P_2 = (t\gamma - \beta\gamma x'^1, x'^2, x'^3)$$

They are infact seperated in time by $\beta\gamma\Delta x$ where Δx is the spatial difference between the events in S.

7.8 Addition of Velocities in Special Relativity

Consider two reference frames S and S' moving with relative speed v along the x-axis. Suppose that the motion of a body in S is described by $\vec{x}(t) = \vec{u}t + \vec{x}_{initial}$. By transforming $\vec{x}$ and t to S', we again get another linear function

$$\vec{x}'(t') = \vec{u}'t' + \vec{x}'_{initial} \tag{7.3}$$

We want to find $\vec{u}'$ in terms of $\vec{u}$ and v. Without loss of generality, we can assume that $\vec{x}_{initial} = 0$. Indeed, if it is not, we can just rebase the coordinate system of S to let the origin coincide with $\vec{x}_{initial}$. By the Lorentz transformation, we have that

$$\begin{aligned} ct' &= \gamma(ct - \beta x) = \gamma(c - \beta u_x)t \\ x' &= \gamma(x - \beta ct) = \gamma(u_x - \beta c)t \\ y' &= y = u_y t \\ z' &= z = u_z t \end{aligned}$$

Now, using the Lorentz transformation to express t in terms of t', we get

$$\begin{aligned} x' &= \gamma(u_x - v)\frac{ct'}{\gamma(c - \beta u_x)} \\ y' &= u_y \frac{ct'}{\gamma(c - \beta u_x)} \\ z' &= u_z \frac{ct'}{\gamma(c - \beta u_x)} \end{aligned}$$

Comparing with Equation (7.3), it follows that

$$\begin{aligned} u'_x &= \frac{u_x - v}{1 - \frac{vu_x}{c^2}} \\ u'_y &= \frac{u_y\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{vu_x}{c}} \\ u'_y &= \frac{u_z\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{vu_x}{c}} \end{aligned}$$

Example 7.8.1. *Consider an object emitting light homogeneously in all directions moving at a speed v. We focus on the rays of light that are moving directly up and down with velocities $u = (0, \pm, 0)$. Applying the Lorentz transformation, we see that*

$$u'_x = -v, u'_y = \pm\sqrt{c^2 - v^2}$$

We see that $|\vec{u}| = c$ so the speed of light is indeed constant but it is now directed along a different vector. To find the angle this vector makes with the x-axis, note that

$$\begin{aligned} -v &= |\vec{u}|\,|\vec{e}_x|\cos\theta = c\cos\theta \\ \implies \cos\theta &= \frac{-v}{c} \end{aligned}$$

We see that for v very close to c, the angle θ is very small. Hence half of all the radiation will be emitted inside a tiny cone in front of the moving particle.

7.9 Relativistic Dynamics

Proposition 7.9.1. *Consider a particle moving at a velocity $\vec{v}$. Then the quantity*

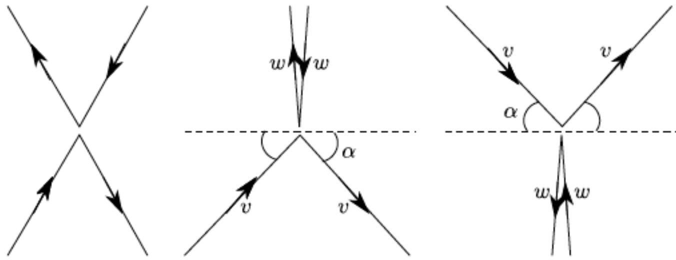
$$\vec{p} = m(|\vec{v}|)\vec{v}$$

is conserved and is called the ***relativistic momentum****. Here*

$$m(|\vec{v}|) = m_0\gamma$$

is the ***relativistic mass****.*

Proof. We only derive the formula for relativistic mass. Consider two identical particles moving towards each other with exactly equal velocities. Their total momentum must be zero. After the collision, their directions of motion must be exactly opposite each other as implied by conservation of momentum. Note that they may still scatter at an angle.This situation is the one described by Frame 1 in the picture below:



Now consider a second frame moving along the x-axis with speed equal to the x component of the first particle's velocity. In such a frame, the x coordinate of the first particle will not change. We denote the speed of this particle by w and the other particle's speed by w.
We again consider another frame now moving along the x-axis with speed equal to that of the x component of the velocity of the second particle. By symmetry, we have that the first particle's speed is v and the second one's w.
We first note that the speed of Frame 3 relative to Frame 2 is $v_x = vcos\alpha$. Using the addition of velocities formulae, we have that the y components of the velocities of the first particle w (in Frame 2) and $v \sin\alpha$ (in Frame 3) are related by

$$w = v \sin\alpha \frac{\sqrt{1 - \frac{v^2 cos^2\alpha}{c^2}}}{1 - \frac{v^2 cos^2\alpha}{c^2}} = \frac{v \sin\alpha}{\sqrt{1 - \frac{v^2 cos^2\alpha}{c^2}}}$$

We hence have a relation between w and v. Now we require that the relativistic momentum is conserved in Frame 2. Along the y-axis, the conservation should give

$$2m(v) v \sin\alpha = 2m(w) w$$
$$\implies \frac{m(w)}{m(v)} = \frac{v \sin\alpha}{w} = \sqrt{1 - \frac{v^2 \cos^2\alpha}{c^2}}$$

□

Since the scattering angle is a free parameter we can, and will, choose it to be very small. This means that in Frame 1, both particles travel almost parallel to the x-axis and barely touch. In Frame 2, this will result in a tiny angle α and very small w. We can take the limit as $\alpha, w \to 0$ and we arrive at

$$m(v) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Proposition 7.9.2. *Consider a particle moving at a velocity $\vec{v}$. Then its kinetic energy is given by*

$$E(\vec{v}) = m(|\vec{v}|)c^2$$

Proof. By Newton's Second Law, we have that

$$\vec{F} = \frac{d\vec{p}}{dt}$$

We know that the change to the energy of the system is given by the work done $\vec{F} \cdot \vec{r}$. Therefore

$$\begin{aligned} dE &= \vec{F} \cdot \vec{r} \\ &= \frac{d\vec{p} \cdot d\vec{r}}{dt} \end{aligned}$$

but $\frac{d\vec{r}}{dt}$ is just the velocity of the particle v. Hence we get

$$\begin{aligned} dE &= d\vec{p} \cdot \vec{v} \\ &= d[m(|\vec{v}|)\vec{v}] \cdot \vec{v} \\ &= dm|\vec{v}|^2 + m(|\vec{v}|)d\vec{v} \cdot \vec{v} \\ &= dm|\vec{v}|^2 + m(|\vec{v}|)\frac{1}{2}d[|\vec{v}|^2] \end{aligned} \tag{7.4}$$

Now using the fact that $v^2 = c^2\left(1 - \frac{m_0^2}{m^2}\right)$ and taking the differential, it follows that

$$dv^2 = c^2 m_0^2 \frac{2dm}{m^3}$$

Inserting this into Equation (7.4) yields

$$dE = dmc^2 \left(1 - \frac{m_0^2}{m^2} + c^2 m_0^2 \frac{dm}{m^2} = dmc^2\right)$$

Which leaves us with

$$E(v) = m(v)c^2 + k$$

for some constant k. Since the energy is always defined up to a constant, we can set it to zero to arrive at the desired result. □

Remark. *For $v << c$, we can Taylor expand to get:*

$$E(v) = m_0 c^2 + \frac{1}{2} m v^2 + \mathcal{O}(\frac{v^4}{c^2 0})$$

Again, the constant can be ignored and we recover the classical kinetic energy.

Chapter 8

Lorentz Group

Definition 8.0.3. *We define the* ***space-time metric*** *(or interval) to be the following quantity*

$$ds^2 = (ct)^2 - x^2 - y^2 - z^2$$

It is the measure of distance between two points in space-time.

Proposition 8.0.4. *Consider two frames of reference S and S' moving with relative speed v with coordinates (t, x, y, z) and (t', x', y', z'). Then the Lorentz transformation between S and S' leaves the space-time interval invariant.*

Proof. We need to show that $(ct)^2 - x^2 - y^2 - z^2 = (ct')^2 - x'^2 - y'^2 - z'^2$. We shall prove this using the rapidity parametrisation. Let θ be the rapidity variable such that $\tanh\theta = \frac{v}{c}$. Then $\cosh\theta = \gamma$ and $\sinh\theta = \beta\gamma$. Then we have that

$$\begin{aligned}
(ct')^2 - x'^2 - y'^2 - z'^2 &= (ct\cosh\theta - x\sinh\theta)^2 - (-ct\sinh\theta + x\cosh\theta)^2 - y^2 - z^2 \\
&= c^2t^2\cosh^2\theta - 2xct\cosh\theta\sinh\theta + x^2\sinh\theta \\
&- c^2t^2\sinh\theta + 2xct\sinh\theta\cosh\theta - x^2\cosh^2\theta - y^2 - z^2 \\
&= c^2t^2\cosh^2\theta + x^2\sinh\theta - c^2t^2\sinh\theta - x^2\cosh^2\theta - y^2 - z^2 \\
&= c^2t^2(\cosh^2\theta - \sinh^2\theta) + x^2(-\cosh^2\theta + \sinh^2\theta) - y^2 - z^2 \\
&= c^2t^2 - x^2 - y^2 - z^2
\end{aligned}$$

as required. □

Proposition 8.0.5. *Consider the set*

$$\mathcal{L} = \left\{\Lambda \,|\, \Lambda^T \eta \Lambda = \eta\right\}$$

where

$$\eta \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Then $\mathcal{L}$ is a group with the composition of transformations as its binary operation. It is referred to as the ***Lorentz Group****.*

Proof. We first show that $\mathcal{L}$ contains the identity transformation. This is trivial as indeed $1^T \eta 1 = \eta$.
Now let $\Lambda \in \mathcal{L}$. We have to find a $\Lambda^{-1} \in \mathcal{L}$ such that $\Lambda\Lambda^{-1} = 1$. Since $\eta\eta = 1$, it follows that $\eta\Lambda^T \eta\Lambda = 1$. Hence $\Lambda^{-1} = \eta\Lambda^T \eta$.
Finally, we show that given $\Lambda_1, \Lambda_2 \in \mathcal{L}$, $\Lambda_1\Lambda_2 \in \mathcal{L}$. By definition we have that $\Lambda_1^T \eta\Lambda_1 = \eta$ and $\Lambda_2^T \eta\Lambda_2 = \eta$. Obviously we have that

$$\begin{aligned} &\Lambda_2^T \eta \Lambda_2 = \eta \\ &\implies \Lambda_2^T \Lambda_1^T \eta \Lambda_1 \Lambda_2 = \eta \\ &\implies (\Lambda_1\Lambda_2)^T \eta (\Lambda_1\Lambda_2) = \eta \end{aligned}$$

hence $\Lambda_1\Lambda_2 \in \mathcal{L}$. □

Remark. *We note that the condition $\Lambda^T \eta \Lambda = \eta$ is equivalent to Λ preserving the space-time interval. Indeed*

$$\vec{x}'^T \eta \vec{x}' = (\Lambda\vec{x})^T \eta (\Lambda\vec{x}) = \vec{x}^T \Lambda^T \eta \Lambda \vec{x} = \vec{x}^T (\Lambda^T \eta \Lambda) \vec{x} = \vec{x}^T \eta \vec{x}$$

Hence $\Lambda \in \mathcal{L}$ if and only if Λ preserves the space-time interval.

8.1 Lorentz Tensors

Convention 8.1.1. *We will use the following conventions:*

- *Lorentz transformation matrix -* $\Lambda^\alpha{}_\beta$

- *Inverse of transpose of Lorentz transformation matrix -* ${\Lambda_\alpha}^\beta$
- *Constant 4x4 matrix* η *-* $\eta_{\alpha\beta}$ *or* $\eta^{\alpha\beta}$
- *Constant 4x4 identity matrix -* δ^α_β

where α *refers to a row element and* β *refers to a column element.*

Definition 8.1.2. *Consider a vector* $\vec{x}$ *that transforms under the Lorentz transformation as follows*

$$\vec{x}' = \Lambda\vec{x}$$

Then we say that $\vec{x}$ *is a* ***contravariant Lorentz vector****. The indices of* $\vec{x}$ *are denoted by upper Greek indices* x^μ*. The transformation rule is thus written*

$$x'^\alpha = {\Lambda^\alpha}_\beta x^\beta$$

Definition 8.1.3. *Consider a vector* $\vec{x}$ *that transforms under the Lorentz transformation as follows*

$$\vec{x}' = (\Lambda^T)^{-1}\vec{x}$$

Then we say that $\vec{x}$ *is a* ***covariant Lorentz vector****. The indices of* $\vec{x}$ *are denoted by lower Greek indices* x_μ*. The transformation rule is thus written*

$$x'_\alpha = {\Lambda_\alpha}^\beta x_\beta$$

Example 8.1.4. *The vector*

$$\left(\frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

is covariant. Indeed,

$$\left(\frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{c\partial t'}{c\partial t}\frac{\partial}{c\partial t'} + \frac{\partial x'}{c\partial t}\frac{\partial}{\partial x'} + \frac{\partial y'}{c\partial t}\frac{\partial}{\partial y'} + \frac{\partial z'}{c\partial t}\frac{\partial}{\partial z'}, \ldots\right)$$

Now using $ct' = {\Lambda^0}_0 ct + {\Lambda^0}_1 x + {\Lambda^0}_2 y + {\Lambda^0}_3 z$ *and the other Lorentz transformations for the other coordinates, it follows that*

$$\begin{aligned}\left(\frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) &= \left({\Lambda^0}_0\frac{\partial}{c\partial t'} + {\Lambda^1}_0\frac{\partial}{\partial x'} + {\Lambda^2}_0\frac{\partial}{\partial y'} + {\Lambda^3}_0\frac{\partial}{\partial z'}, \ldots\right) \\ &= \Lambda^T\left(\frac{\partial}{c\partial t'}, \frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\right)\end{aligned}$$

Multiplying on the left by $(\Lambda^T)^{-1}$*, we arrive at the desired result.*

Remark. *In order to conserve Einstein's summation convention in tensor index notation, we require the following two rules:*

- *Dummy indices must be in opposite positions -* $v^\mu u_\mu$
- *Positions of free indices must be the same on both sides of the equation -* $v^\mu = u^\mu$

Example 8.1.5. *As a consequence of the new notation, we have the following identity*

$$\delta^\alpha_\beta = \Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta$$

8.2 General Lorentz Tensor

Definition 8.2.1. *The **general Lorentz tensor** is an object with an arbitrary number of indices* $T_{\alpha_1}{}^{\beta_1}{}_{\alpha_2}{}^{\beta_2}{}_{\dots}{}^{\dots}$. *They transform according to the position of the corresponding indices.*

We can easily build new tensors out of old ones. For example, we can convert a contravariant vector u^μ to a covariant one u_ν as follows

$$u_\nu = \eta_{\nu\mu} u^\mu$$

We can check that u_ν is indeed covariant:

$$u'_\nu = \eta_{\nu\mu} u'^\mu = \eta_{\nu\mu} \Lambda^\mu{}_\alpha u^\alpha = \eta_{\nu\mu} \Lambda^\mu{}_\alpha \eta^{\alpha\beta} u_\beta = \Lambda_\nu{}^\beta u_\beta$$

We can always raise or lower indices of any Lorentz tensor using η. For example, consider the tensor $T^\mu{}_\nu$ which is a (contra,co)-variant tensor. $T^\mu{}_\nu$ can be transformed into 3 other tensors (we pack $T^\mu{}_\nu$ into a 4×4 matrix for visualisation purposes):

- $T^{\mu\nu} = T^\mu{}_\alpha \eta^{\alpha\nu}$ - equivalent to the matrix $T\eta$
- $T_{\mu\nu} = \eta_{\mu\alpha} T^\alpha{}_\nu$ - equivalent to the matrix ηT
- $T_\mu{}^\nu = \eta_{\mu\alpha} T^\alpha{}_\beta \eta^{\beta\nu}$ - equivalent to the matrix $\eta T \eta$

We can also create new tensors by multiplying two tensors. For example $a^\mu b_\nu$.
We can also contract indices of a Lorentz tensor to reduce the rank of a tensor by two. For example, consider the rank 3 tensor $T^{\alpha\beta}{}_\gamma$. We can generate a new rank 1 tensor by contracting the indices as follows: $u^\alpha = T^{\alpha\beta}{}_\beta$.

Definition 8.2.2. *Consider the two vectors $\vec{a}$ and $\vec{b}$. We define the* ***inner product*** *of $\vec{a}$ and $\vec{b}$ to be the product $a^\mu b_\mu$.*

Proposition 8.2.3. *Consider the two vectors $\vec{a}$ and $\vec{b}$. Then their inner product is Lorentz invariant.*

Proof. Let $\vec{\alpha}'$ and $\vec{\beta}'$ be the vectors $\vec{a}$ and $\vec{b}$ under the action of the Lorentz transformation. Then

$$a'^\mu b'_\mu = (\Lambda^\mu{}_\alpha a^\alpha)(\Lambda_\mu{}^\beta b_\beta) = a^\alpha \Lambda^\mu{}_\alpha \Lambda_\mu{}^\beta b_\beta = a^\alpha \delta^\beta_\alpha b_\beta = a^\alpha b_\alpha$$

as required. □

Example 8.2.4. *There are many examples of inner products which gives us important Lorentz invariants:*

- $\delta^\mu x_\mu = 4$
- $x^\mu x_\mu = x^\mu x^\nu \eta_{\mu\nu} = (x^0)^2 - (x^1)^2 - (x^3)^2 = (ct)^2 - x^2 - y^2 - z^2$ *- this is just the space-time interval which have shown to be invariant in the previous section.*
- $\delta^\mu \delta_\mu f = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) f$ *- this is just the wave operator. From this, we can see that the Maxwell's equations in the vacuum are Lorentz invariant.*

Definition 8.2.5. *Let a_μ be a covariant (or contravariant) vector. We have the following three classifications of a_μ based on their inner product:*

- ***Light-like vectors*** *- $a_\mu a^\mu = 0$. For example, take $a_\mu = (1, 1, 0, 0)$. Then $a^\mu = a_\nu \eta^{\nu\mu} = (1, -1, 0, 0)$ and thus $a_\mu a^\mu = 0$.*
- ***Time-like vectors*** *- $a_\mu a^\mu > 0$. For example, take $a_\mu = (1, 0, 0, 0)$. (this vector connects two events which are causually connected - the second event lies within the lightcone of the first event)*

- ***Space-like vectors*** - $a_\mu a^\mu < 0$. *For example, take* $a_\mu = (0,1,0,0)$. *(this vector connects two events which are causually disconnected - the second event lies outside the lightcone of the first event)*

Example 8.2.6. *Lorentz invariant tensors do not have to be rank 0. One example is the* $\eta^{\mu\nu}$ *tensor. It has the non-trivial transformation property* $\eta'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta^{\alpha\beta}$. *But* $\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta^{\alpha\beta} = \eta^{\mu\nu}$ *by the definition of a Lorentz tensor.*

Example 8.2.7. *The 4-dimensional Levi-Civita (epsilon) tensor* $\epsilon^{\alpha\beta\mu\nu}$ *is a* ***pseudo-invariant*** *tensor.The epsilon tensor transforms as follows:*

$$\epsilon'^{\alpha'\beta'\mu'\nu'} = \Lambda^{\alpha'}{}_\alpha \Lambda^{\beta'}{}_\beta \Lambda^{\mu'}{}_\mu \Lambda^{\nu'}{}_\nu \epsilon^{\alpha\beta\mu\nu}$$

In particular for

$$\epsilon'^{0123} = \Lambda^0{}_\alpha \Lambda^1{}_\beta \Lambda^2{}_\mu \Lambda^3{}_\nu \epsilon^{\alpha\beta\mu\nu}$$

On the right hand side, we have to sum over all the permutations of 0123 *thus we have* $4! = 24$ *terms on the right hand side of the form*

$$\Lambda^0{}_0 \Lambda^1{}_1 \Lambda^2{}_2 \Lambda^3{}_3 - \Lambda^0{}_1 \Lambda^1{}_1 \Lambda^2{}_2 \Lambda^3{}_3 + \dots$$

We can recognise this as the determinant of the 4×4 *matrix* Λ. *Hence*

$$\epsilon'^{0123} = \det \Lambda$$

We can apply some linear algebra to $\Lambda^T \eta \Lambda = \eta$ to see that $(\det \Lambda)^2 = 1$. Thus $\det \Lambda = \pm 1$. We see that ϵ keeps its sign for half of all transformations and changes it for the other half. We call ϵ a pseudo-invariant tensor - it is invariant up to a modulo.